

## **N=2 gauge theories, instanton moduli spaces and geometric representation theory**

**Citation for published version:**

Szabo, RJ 2016, 'N=2 gauge theories, instanton moduli spaces and geometric representation theory', *Journal of Geometry and Physics*, vol. 109, pp. 83-121. <https://doi.org/10.1016/j.geomphys.2015.09.005>

**Digital Object Identifier (DOI):**

[10.1016/j.geomphys.2015.09.005](https://doi.org/10.1016/j.geomphys.2015.09.005)

**Link:**

[Link to publication record in Heriot-Watt Research Portal](#)

**Document Version:**

Peer reviewed version

**Published In:**

Journal of Geometry and Physics

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## Accepted Manuscript

$\mathcal{N} = 2$  gauge theories, instanton moduli spaces and geometric representation theory

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PII: S0393-0440(15)00222-3

DOI: <http://dx.doi.org/10.1016/j.geomphys.2015.09.005>

Reference: GEOPHY 2636

To appear in: *Journal of Geometry and Physics*

Received date: 2 July 2015

Accepted date: 8 September 2015

Please cite this article as: R.J. Szabo,  $\mathcal{N} = 2$  gauge theories, instanton moduli spaces and geometric representation theory, *Journal of Geometry and Physics* (2015), <http://dx.doi.org/10.1016/j.geomphys.2015.09.005>

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# $\mathcal{N} = 2$ gauge theories, instanton moduli spaces and geometric representation theory<sup>1</sup>

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## Abstract

We survey some of the AGT relations between  $\mathcal{N} = 2$  gauge theories in four dimensions and geometric representations of symmetry algebras of two-dimensional conformal field theory on the equivariant cohomology of their instanton moduli spaces. We treat the cases of gauge theories on both flat space and ALE spaces in some detail, and with emphasis on the implications arising from embedding them into supersymmetric theories in six dimensions. Along the way we construct new toric noncommutative ALE spaces using the general theory of complex algebraic deformations of toric varieties, and indicate how to generalise the construction of instanton moduli spaces. We also compute the equivariant partition functions of topologically twisted six-dimensional Yang–Mills theory with maximal supersymmetry in a general  $\Omega$ -background, and use the construction to obtain novel reductions to theories in four dimensions.

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<sup>1</sup>Contribution to the Special Issue “Instanton Counting: Moduli Spaces, Representation Theory and Integrable Systems” of the *Journal of Geometry and Physics*, eds. U. Bruzzo and F. Sala.

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## 1 Introduction and summary

This paper is a survey of some recent mathematical results concerning the ever evolving topic of instanton counting in four-dimensional gauge theories with  $\mathcal{N} = 2$  supersymmetry. For brevity of the exposition we consider only gauge theories without matter fields, with gauge group  $U(N)$ , and in the absence of complex codimension one singularities. Our survey is mostly technical and phrased in mathematical language, presenting details for the most part only where firm mathematical statements can be made. In particular, throughout this paper we exploit the geometric interpretation of instantons as moduli of torsion-free sheaves together with their algebraic realisations as modules over a noncommutative deformation of the pertinent space. We pay special attention to ways in which four-dimensional theories can be realised as reductions of certain supersymmetric theories in six dimensions, and discuss some of their implications; other surveys devoted to the interplay between supersymmetric gauge theories in four and six dimensions, with different emphasis, can be found in [101, 102, 29]. We consider three such six-dimensional perspectives that we now explain in detail as a summary of the contents of the present contribution.

We begin in §2 with a mathematical introduction to topologically twisted  $\mathcal{N} = 2$  gauge theories in four dimensions, which can be defined as a naive dimensional reduction of  $\mathcal{N} = 1$  Yang–Mills theory in six dimensions. After explaining how the principle of supersymmetric localization reduces the partition functions to integrals of certain characteristic classes over the instanton moduli spaces, we consider a particular twisted dimensional reduction from six dimensions which defines Nekrasov’s  $\Omega$ -deformation of the four-dimensional  $\mathcal{N} = 2$  gauge theory [83]. This recasts the partition function as a generating function for certain integrals in equivariant cohomology, to which powerful localization techniques can be applied. Our main examples in this paper concern gauge theories on  $\mathbb{C}^2$ , its cyclic orbifolds, and its resolutions, i.e. ALE spaces. We provide a detailed description of the moduli spaces of instantons on ALE spaces, and the computations of their partition functions, elucidating various details. We also report on preliminary steps towards a new reformulation of the instanton partition functions in terms of contributions from modules over a toric noncommutative deformation of ALE spaces, extending the constructions of [34, 35] which dealt with gauge theories on  $\mathbb{C}^2$ ; this

provides rigorous justification to some heuristic calculations in the literature within the context of noncommutative gauge theory, and moreover produces new instanton moduli spaces which are (commutative) deformations of those of the classical case.

In §3 we consider the four-dimensional  $\mathcal{N} = 2$  gauge theory as a reduction of the elusive  $\mathcal{N} = (2, 0)$  theory in six dimensions which is the trigger for the celebrated duality between gauge theories in four dimensions and conformal field theory in two dimensions that was discovered by Alday, Gaiotto and Tachikawa [3]. We survey some of the results that have been obtained in this direction thus far, focusing again on instances where precise mathematical statements can be made concerning the geometric realisations of the equivariant cohomology of instanton moduli spaces as modules for  $\mathcal{W}$ -algebras. We also summarise some of the results that relate these representations to the spectra of quantum integrable systems underlying the  $\mathcal{N} = 2$  gauge theories. However, we stress that this article is *not* meant to serve as a review of the AGT correspondence, and it is beyond its scope to survey all aspects and literature on the subject. A review of the AGT correspondence with emphasis on physical aspects can be found in [103] where a more exhaustive survey and list of references is presented, together with a pedagogical introduction to instanton moduli spaces and the computations of equivariant partition functions; another recent pedagogical review for physicists with different emphasis and vast literature survey can be found in [61].

In §4 we study the cohomological gauge theory in six dimensions with maximal ( $\mathcal{N} = 2$ ) supersymmetry; in a certain sense, that we discuss, it can be regarded as sitting somewhere between the topologically twisted pure  $\mathcal{N} = 2$  gauge theory and the cohomological gauge theory in four dimensions with maximal ( $\mathcal{N} = 4$ ) supersymmetry. Its partition function localises to a weighted sum over generalised instantons which calculates the Donaldson–Thomas theory of toric Calabi–Yau threefolds. Here we compute the Coulomb branch partition function for an arbitrary  $\Omega$ -deformation of  $\mathbb{C}^3$ , extending the calculation of [31] which considered only the Calabi–Yau specialisation; similar treatments in the framework of equivariant K-theory can be found in [84, 7]. We use this general deformation to observe two reductions of the six-dimensional  $\mathcal{N} = 2$  gauge theory to a four-dimensional theory; we do not understand fully the implications of these reductions yet, but we feel they are novel and worthy of mention. First, we show that instantons in four dimensions can be embedded as generalised instantons in six dimensions which are invariant under a certain  $\mathbb{C}^\times$ -action; this is analogous to the construction of [56] which shows that the moduli space of vortices in two-dimensional  $\mathcal{N} = (4, 4)$  gauge theories can be embedded as a  $U(1)$ -invariant subspace of the instanton moduli space. When the gauge theory is viewed as the worldvolume theory of D6-branes (as is required in the definition of the  $\Omega$ -deformation in this case), this reduction seems to be consistent with previously noted reductions in the literature; however, we do not understand what six-dimensional theory would reduce exactly to the  $\mathcal{N} = 2$  gauge theories in four dimensions. Second, in the general  $\Omega$ -background we can consider the analog of the Nekrasov–Shatashvili limit [88] which clearly reduces the gauge theory to some exactly solvable four-dimensional theory whose properties remain to be understood.

We conclude with three appendices at the end of the paper which contain pertinent technical details used in the main text. In §A we summarise the relevant combinatorial background on Young diagrams and partitions which are used throughout this paper; the standard reference is the book [6] in which further details and concepts may be found. In §B we survey aspects from the theory of symmetric functions; the standard reference is the book [70] which can be consulted for further general details, though we give a fairly detailed exposition of the Uglov functions as they are perhaps the less familiar class of symmetric functions and are particularly relevant to the discussion of §3. Finally, in §C we summarise some standard results about  $\mathcal{W}$ -algebras; further particulars can be found in the review [21].

## 2 $\mathcal{N} = 2$ gauge theories in four dimensions

In this section we study supersymmetric gauge theories in four dimensions using their relation to the cohomology of moduli spaces of torsion-free sheaves. Our main focus is on formulating instanton counting problems. In particular, we provide a fairly detailed account of moduli of instantons on ALE spaces, and the connections between noncommutative instantons and framed sheaves. We elucidate details of some calculations below, while the material in the last two subsections is new.

### 2.1 Supersymmetric localization

The main player in this paper is the moduli space of finite energy charge  $n$  instantons in gauge theory on a Riemannian four-manifold  $X$  (with Hodge operator  $*$ ). This is the space of gauge equivalence classes of connections on a principal bundle on  $X$ , which are flat at infinity, of second Chern class  $n$  whose curvature  $F$  satisfies the anti-self-duality equation  $F = -*F$ . The modern interest in these moduli varieties is the important role they play in supersymmetric quantum gauge theories. Gauge theories with  $\mathcal{N} = 2$  supersymmetry can be topologically twisted and, via the principle of supersymmetric localization, the formal functional integrals defining their partition functions receive non-trivial contributions from only supersymmetry-preserving configurations. These states are precisely the instanton solutions, and hence the partition function and BPS observables can be rigorously *defined* as finite-dimensional integrals of certain characteristic classes of natural vector bundles over the instanton moduli spaces which are induced by integrating out the bosonic and fermionic field degrees of freedom. We begin by sketching how some of these formal arguments work; see e.g. [36] for a review.

Let  $G$  be a compact semi-simple Lie group with Lie algebra  $\mathfrak{g}$ , and let  $P \rightarrow X$  be a principal  $G$ -bundle. Let  $\mathcal{A}(P)$  be the space of connections  $A$  on  $P$  which are flat at infinity. Since  $\mathcal{A}(P)$  is an affine space, the tangent space  $T_A\mathcal{A}(P)$  at any point  $A \in \mathcal{A}(P)$  can be canonically identified with  $\Omega^1(X, \text{ad } P)$ . The *twisted  $\mathcal{N} = 2$  supersymmetry algebra* is the algebra of odd derivations of the differential graded algebra  $\Omega^\bullet(\mathcal{A}(P))$  of differential forms on  $\mathcal{A}(P)$ ; it is generated by supercharges  $\delta$  with  $\delta^2 = 0$  and carries a super-Lie algebra structure. The space of bosonic fields  $\mathcal{V}(P)$  of twisted  $\mathcal{N} = 2$  gauge theory on  $X$  can be identified by regarding it as the dimensional reduction of six-dimensional  $\mathcal{N} = 1$  supersymmetric Yang-Mills theory on the trivial complex line bundle  $X \times \mathbb{C}$  over  $X$  in the limit where the fiber collapses to a point; by identifying the vertical components of connections in this limit with sections  $\phi \in \Omega^0(X, \text{ad } P)$ , one has

$$\mathcal{V}(P) = \mathcal{A}(P) \times \Omega^0(X, \text{ad } P) .$$

The superpartners of these fields are sections of the cotangent bundle  $T^*\mathcal{V}(P)$ , so that the total field content of the  $\mathcal{N} = 2$  gauge theory on  $X$  lies in the infinite-dimensional space

$$\mathcal{W}(P) = \mathcal{V}(P) \times \Omega^\bullet(\mathcal{V}(P)) .$$

One can define a family of noncommutative associative products on these fields, parameterized locally by points  $p \in X$ , which turns  $\mathcal{W}(P)$  into an algebra that generalizes the structure of a vertex operator algebra in two-dimensional conformal field theory. The supersymmetry generators  $\delta$  act on  $\mathcal{W}(P)$  and so one can define the cohomology  $H^\bullet(\mathcal{W}(P), \delta)$ . The product on fields induces a standard superalgebra structure on  $H^\bullet(\mathcal{W}(P), \delta)$ , which gives the *chiral ring* of the  $\mathcal{N} = 2$  gauge theory on  $X$ .

Let  $\mathcal{G}(P)$  be the group of automorphisms of  $P$  which are trivial at infinity; its Lie algebra can be identified with the space of sections  $\Omega^0(X, \text{ad } P)$ . Let

$$\mathfrak{M}_{G,P}(X) := \{(A, \phi) \in \mathcal{V}(P) \mid F^+ = 0, \nabla_A \phi = 0\} / \mathcal{G}(P)$$

denote the moduli space of pairs of anti-self-dual connections and covariantly constant sections, where  $F^+ := \frac{1}{2}(F + *F)$  denotes the self-dual part of the curvature two-form  $F \in \Omega^2(X, \text{ad } P)$  of  $A$  and  $\nabla_A$  is the covariant derivative associated to  $A$ . There is an infinite rank vector bundle  $\pi : \mathcal{E} \rightarrow \mathcal{V}(P)/\mathcal{G}(P)$  given by

$$\mathcal{E} := \mathcal{V}(P) \times_{\mathcal{G}(P)} (\Omega^{2,+}(X, \text{ad } P) \oplus \Omega^1(X, \text{ad } P))$$

where the superscript  $+$  denotes the self-dual part of a two-form, on which the Hodge operator  $*$  acts as the identity; it is associated to the principal  $\mathcal{G}(P)$ -bundle  $\mathcal{V}(P) \rightarrow \mathcal{V}(P)/\mathcal{G}(P)$ . Let  $\text{Th}(\mathcal{E}) := \pi_*^{-1}(1) \in H^\bullet(\mathcal{E})$  be the *Thom class* of  $\mathcal{E}$ , where  $\pi_* : H^\bullet(\mathcal{E}) \rightarrow H^\bullet(\mathcal{V}(P)/\mathcal{G}(P))$  is the Thom isomorphism. The vector bundle  $\mathcal{E}$  has a natural section

$$s(A, \phi) = (F^+, \nabla_A \phi)$$

whose zero locus coincides with the moduli space  $\mathfrak{M}_{G,P}(X)$ . Then the pullback of the Thom class  $s^*(\text{Th}(\mathcal{E}))$  is a closed form representing the Euler class  $\text{Eu}(\mathcal{E})$  of the vector bundle  $\mathcal{E}$ . It obeys the localization property

$$\int_{\mathcal{V}(P)/\mathcal{G}(P)} s^*(\text{Th}(\mathcal{E})) \wedge \mathcal{O} = \int_{\mathfrak{M}_{G,P}(X)} i^* \mathcal{O}$$

for any gauge invariant observable  $\mathcal{O}$  on  $\mathcal{V}(P)$ , i.e.  $s^*(\text{Th}(\mathcal{E}))$  is the Poincaré dual to the embedding  $i : \mathfrak{M}_{G,P}(X) \hookrightarrow \mathcal{V}(P)/\mathcal{G}(P)$ .

The partition function of cohomological Yang–Mills theory can be interpreted geometrically as constructing the Mathai–Quillen representative of the Thom class  $\text{Th}(\mathcal{E})$ , which is accomplished in the twisted  $\mathcal{N} = 2$  gauge theory on  $X$  via supersymmetric localization; this is the cohomological gauge theory that gives rise to the Donaldson invariants of four-manifolds [108]. Using standard localization arguments applied to the  $\mathcal{G}(P)$ -equivariant cohomology of the space of fields  $\mathcal{W}(P)$ , the weighted integration over the domain  $\mathcal{W}(P)$  localizes to an integral over the  $\delta$ -fixed points, i.e. to BPS observables representing cohomology classes in the chiral ring  $H^\bullet(\mathcal{W}(P), \delta)$ , which under favourable conditions consists of irreducible connections with  $\phi = 0$ . By taking a weighted sum over all topological classes of principal  $G$ -bundles  $P \rightarrow X$ , the partition function of the  $\mathcal{N} = 2$  gauge theory on  $X$  is then given *exactly* by

$$\mathcal{Z}_X(\mathbf{q}) = \sum_{n=0}^{\infty} \mathbf{q}^n \int_{\mathfrak{M}_{G,n}(X)} \mathcal{Z}(P \rightarrow X) ,$$

where  $\mathbf{q} = e^{2\pi i \tau}$  with  $\tau$  the complexified gauge coupling constant,  $\mathfrak{M}_{G,n}(X)$  is the moduli space of anti-self-dual  $G$ -connections on a principal  $G$ -bundle  $P \rightarrow M$  with  $c_2(P) = n$ , and the characteristic class  $\mathcal{Z}(P \rightarrow X)$  is a topological invariant depending only on the smooth structure of  $X$ . The instanton moduli space  $\mathfrak{M}_{G,n}(X)$  has real dimension  $4hn$ , where  $h$  is the dual Coxeter number of  $G$ . However, even though in some instances the classes  $\mathcal{Z}(P \rightarrow X)$  are given by tractable expressions, it is not always the case that the moduli space integrals can be evaluated explicitly.

In this paper we are interested in cases where  $X$  is a complex surface and the structure group  $G$  is the unitary group  $U(N)$ . By the seminal work of Donaldson [40], which translated the problem into the setting of algebraic geometry,  $U(N)$  instantons correspond to rank  $N$  holomorphic vector bundles on  $X$ . A partial compactification which serves as a resolution of singularities of this moduli space is provided by the moduli space  $\mathfrak{M}_{N,n}(X)$  of rank  $N$  torsion-free sheaves  $\mathcal{E}$  on a compactification  $\overline{X} = X \cup D_\infty$  with  $c_2(\mathcal{E}) = n$  and a framing isomorphism (trivialization)  $\phi_\mathcal{E} : \mathcal{E}|_{D_\infty} \rightarrow \mathcal{O}_{D_\infty}^{\oplus r}$  over the compactification divisor  $D_\infty$ . The moduli integrals are still somewhat tricky to handle, and need not even be well-defined due to the non-compactness of  $\mathfrak{M}_{N,n}(X)$ . This problem was resolved



in the seminal work of Nekrasov [83] who showed that with a further deformation of the gauge theory, called  $\Omega$ -deformation, one can make mathematical sense of these moduli space integrals by regarding them as integrations in equivariant cohomology. Nekrasov's theory applies to the instances where  $X$  is a *toric* surface, i.e.  $X$  carries an action of the algebraic torus  $(\mathbb{C}^\times)^2$  which is associated to a fan in  $\mathbb{R}^2$ ; in this case there are extra supercharges and, in addition to the gauge group, the equivariance with respect to the lift of the torus action on  $X$  to the space of fields is a powerful tool for analysing the  $N = 2$  gauge theory.

## 2.2 $N = 2$ gauge theory on $\mathbb{C}^2$

We begin with the case where  $X$  is the complex affine space  $\mathbb{C}^2$ . Then  $\overline{X}$  is the projective plane  $\mathbb{P}^2 = \mathbb{C}^2 \cup \ell_\infty$  with  $D_\infty = \ell_\infty \cong \mathbb{P}^1$  a projective line, and the fine moduli space  $\mathfrak{M}_{N,n}(\mathbb{C}^2)$  is a smooth quasi-projective complex algebraic variety of dimension  $2Nn$ . In the rank one case  $N = 1$ ,  $\mathfrak{M}_{1,n}(\mathbb{C}^2) \cong \text{Hilb}^n(\mathbb{C}^2)$  is the Hilbert scheme of  $n$  points in the plane  $\mathbb{C}^2$  which is the moduli space parameterizing ideals in the polynomial ring  $\mathbb{C}[z_1, z_2] = \mathcal{O}_{\mathbb{C}^2}$  of codimension  $n$ . The larger family of moduli varieties  $\mathfrak{M}_{N,n}(\mathbb{C}^2)$  of torsion-free  $\mathbb{C}[z_1, z_2]$ -modules of rank  $N > 1$  are in this sense higher rank generalizations of Hilbert schemes.

The moduli space  $\mathfrak{M}_{N,n}(\mathbb{C}^2)$  arises in other settings. For example, it is the moduli space of  $n$  D0-branes bound to  $N$  D4-branes in Type IIA string theory [41]. It is also the moduli space of  $U(N)$  noncommutative instantons on a Moyal deformation of Euclidean space  $\mathbb{R}^4$  [87]. In this paper we shall frequently exploit these convenient realisations of instanton moduli as torsion-free modules over a noncommutative deformation of the algebra  $\mathbb{C}[z_1, z_2]$  [63, 34, 35].

The maximal torus  $\mathbb{T}^2 := \mathbb{C}^\times \times \mathbb{C}^\times$  of the group  $GL(2, \mathbb{C})$  acts by complex rotations of  $\mathbb{C}^2$ , while the group of constant gauge transformations  $GL(N, \mathbb{C})$  acts on  $\mathfrak{M}_{N,n}(\mathbb{C}^2)$  by rotating the framing at infinity  $\phi_\varepsilon \mapsto g \circ \phi_\varepsilon$ . Framed instantons are invariant only under local rotations, so we consider the  $\Omega$ -deformed gauge theory in which rotational invariance holds only up to gauge equivalence. This implies that the parameters of the framing rotations are given by the expectation values of the scalar field  $\phi$  in the  $N = 2$  vector multiplet ("Higgs vevs"), which can always be rotated into the Cartan subalgebra of  $\mathfrak{gl}_N$ . Correspondingly, the torus  $\widetilde{\mathbb{T}} = \mathbb{T}^2 \times (\mathbb{C}^\times)^N$  acts on the moduli space  $\mathfrak{M}_{N,n}(\mathbb{C}^2)$ .

The  $\Omega$ -deformed  $N = 2$  gauge theory on  $\mathbb{C}^2$  can be defined as the twisted dimensional reduction of six-dimensional  $N = 1$  gauge theory on the total space of a flat  $\mathbb{C}^2$ -bundle  $M_{\tau_0} \rightarrow T^2$  in the limit where the real two-torus  $T^2$ , of complex structure modulus  $\tau_0$ , collapses to a point; the total space of the affine bundle  $M_{\tau_0}$  is defined as the quotient of  $\mathbb{C}^2 \times \mathbb{C}$  by the  $\mathbb{Z}^2$ -action

$$(n_1, n_2) \triangleright (z_1, z_2, w) = (t_1^{n_1} z_1, t_2^{n_2} z_2, w + (n_1 + \tau_0 n_2))$$

where  $(n_1, n_2) \in \mathbb{Z}^2$ ,  $(z_1, z_2) \in \mathbb{C}^2$ ,  $w \in \mathbb{C}$ , and  $(t_1, t_2) \in \mathbb{T}^2$ . We further introduce a flat  $\mathbb{T}^2$ -bundle over  $T^2$ ; in the collapsing limit, fields of the gauge theory which are charged under the  $SL(2, \mathbb{C})$  R-symmetry group are sections of the pullback of this bundle over  $M_{\tau_0}$ . Let  $(\epsilon_1, \epsilon_2)$  denote its first Chern class, so that  $(t_1, t_2) = (e^{\epsilon_1}, e^{\epsilon_2})$  parameterize the holonomy of a flat connection on the  $\mathbb{T}^2$ -bundle, and let  $\vec{a} = (a_1, \dots, a_N)$  denote the Higgs vevs of the complex scalar field  $\phi$  that is induced from the horizontal components of the six-dimensional gauge connection on  $M_{\tau_0} \rightarrow T^2$ .

The definition of the  $\Omega$ -background can be formalised into the setting of toric geometry, which we shall need in the following in order to generalize it to other spaces. Let  $L$  be a lattice of rank two, and let  $L^* = \text{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$  be the dual lattice; the dual pairing between lattices is denoted  $\langle -, - \rangle : L^* \otimes_{\mathbb{Z}} L \rightarrow \mathbb{Z}$ . Let  $\mathbb{T}^2 = L \otimes_{\mathbb{Z}} \mathbb{C}^\times$  be the associated complex torus of dimension two over  $\mathbb{C}$ . Upon fixing a  $\mathbb{Z}$ -basis  $\{e_1, e_2\}$  for  $L$ , with corresponding dual basis  $\{e_1^*, e_2^*\}$  for  $L^*$ , one has  $L \cong \mathbb{Z} \oplus \mathbb{Z}$ ,  $L^* \cong \mathbb{Z} \oplus \mathbb{Z}$ , and  $\mathbb{T}^2 \cong \mathbb{C}^\times \times \mathbb{C}^\times$ . The torus  $\mathbb{T}^2$  acts on  $\mathbb{C}^2$  in the standard way by scaling



$t \triangleright (z_1, z_2) = (t_1 z_1, t_2 z_2)$  for  $t = e_1 \otimes t_1 + e_2 \otimes t_2 \in \mathbb{T}^2$ . Then the pair  $(\epsilon_1, \epsilon_2)$  are the equivariant parameters which generate the coefficient ring  $H_{\mathbb{T}^2}^2(\text{pt}, \mathbb{Z})$  of  $\mathbb{T}^2$ -equivariant cohomology.

The instanton part of Nekrasov's partition function for pure  $\mathcal{N} = 2$   $U(N)$  gauge theory on  $\mathbb{C}^2$  is the generating function for equivariant volumes of the moduli spaces  $\mathfrak{M}_{N,n}(\mathbb{C}^2)$  defined by

$$\mathcal{Z}_{\mathbb{C}^2}^{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; \mathbf{q}) = \sum_{n=0}^{\infty} \mathbf{q}^n \int_{\mathfrak{M}_{N,n}(\mathbb{C}^2)} 1. \quad (2.1)$$

Here the integrals are understood as pushforward maps to a point in  $\tilde{\mathbb{T}}$ -equivariant cohomology  $H_{\tilde{\mathbb{T}}}^{\bullet}(\mathfrak{M}_{N,n}(\mathbb{C}^2))$ ; they can be computed using the localization theorem in equivariant cohomology which evaluates them as a sum over  $\tilde{\mathbb{T}}$ -fixed points of the moduli spaces, and gives (2.1) as a combinatorial expression in  $\mathbf{q}$  and the equivariant parameters  $(\epsilon_1, \epsilon_2, \vec{a})$ .

The torus fixed points  $\mathfrak{M}_{N,n}(\mathbb{C}^2)^{\tilde{\mathbb{T}}}$  are isolated and parameterized by  $N$ -vectors of Young diagrams  $\vec{Y} = (Y_1, \dots, Y_N)$  of total weight  $|\vec{Y}| = \sum_l |Y_l| = n$  (see §A.1). The localization theorem in equivariant cohomology yields the combinatorial expansion

$$\int_{\mathfrak{M}_{N,n}(\mathbb{C}^2)} 1 = \sum_{|\vec{Y}|=n} \frac{1}{\text{Eu}(T_{\vec{Y}} \mathfrak{M}_{N,n}(\mathbb{C}^2))},$$

and by explicitly computing the equivariant Euler class  $\text{Eu}(T_{\vec{Y}} \mathfrak{M}_{N,n}(\mathbb{C}^2)) \in H_{\tilde{\mathbb{T}}}^{\bullet}(\mathfrak{M}_{N,n}(\mathbb{C}^2))$  of the tangent space at the fixed point  $\vec{Y}$  [45, 25, 82] one finds that the instanton partition function on  $\mathbb{C}^2$  is then given by

$$\mathcal{Z}_{\mathbb{C}^2}^{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; \mathbf{q}) = \sum_{\vec{Y}} \mathbf{q}^{|\vec{Y}|} \prod_{l,l'=1}^N \frac{1}{\tau_{l,l'}^{\vec{Y}}(\epsilon_1, \epsilon_2, a_l - a_{l'})}, \quad (2.2)$$

where

$$\begin{aligned} \tau_{l,l'}^{\vec{Y}}(\epsilon_1, \epsilon_2, a) &= \prod_{s \in Y_l} \left( a - L_{Y_{l'}}(s) \epsilon_1 + (A_{Y_l}(s) + 1) \epsilon_2 \right) \\ &\times \prod_{s' \in Y_{l'}} \left( a + (L_{Y_l}(s') + 1) \epsilon_1 - A_{Y_{l'}}(s') \epsilon_2 \right). \end{aligned}$$

These considerations can be extended to more general  $\mathcal{N} = 2$  quiver gauge theories on  $\mathbb{C}^2$ , wherein the contributions of matter fields are represented by Euler classes of universal vector bundles on  $\mathfrak{M}_{N,n}(\mathbb{C}^2)$ . They are also applicable to the more general  $\mathcal{N} = 2$  gauge theories in four dimensions of class S [48, 50], which we discuss in §3.

In the rank one case  $N = 1$ , the combinatorial series (2.2) can be explicitly summed to give the pure  $\mathcal{N} = 2$  gauge theory partition function as a simple exponential [82, §4]

$$\mathcal{Z}_{\mathbb{C}^2}^{\text{inst}}(\epsilon_1, \epsilon_2; \mathbf{q}) = \exp \left( \frac{\mathbf{q}}{\epsilon_1 \epsilon_2} \right). \quad (2.3)$$

This formula can be derived from the Cauchy–Stanley formula for Jack symmetric functions, see §B.3. See [91, §4] for a direct representation theoretic proof of this expression.

For later comparisons with the partition functions of the six-dimensional  $(2,0)$  theory that we discuss in §3 and of the six-dimensional  $\mathcal{N} = 2$  gauge theory that we consider in §4, we should further multiply the instanton partition function (2.2) by the classical contribution

$$\mathcal{Z}_{\mathbb{C}^2}^{\text{cl}}(\epsilon_1, \epsilon_2, \vec{a}; \mathbf{q}) = \prod_{l=1}^N \mathbf{q}^{-\frac{a_l^2}{2\epsilon_1 \epsilon_2}}, \quad (2.4)$$

and also by the purely perturbative contribution which is  $\mathfrak{q}$ -independent and given by

$$\mathcal{Z}_{\mathbb{C}^2}^{\text{pert}}(\epsilon_1, \epsilon_2, \vec{a}) = \exp \left( - \sum_{l, l'=1}^N \gamma_{\epsilon_1, \epsilon_2}(a_{l'} - a_l; \Lambda) \right). \quad (2.5)$$

Here we defined the Barnes double zeta-function [81, §E.2]

$$\gamma_{\epsilon_1, \epsilon_2}(x; \Lambda) := \lim_{s \rightarrow 0} \frac{d}{ds} \frac{\Lambda^s}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \frac{e^{-tx}}{(1 - e^{t\epsilon_1})(1 - e^{t\epsilon_2})},$$

which is a regularization of the formal expression

$$\gamma_{\epsilon_1, \epsilon_2}(x; \Lambda) = \sum_{m, n \geq 0} \log \left( \frac{x - m\epsilon_1 - n\epsilon_2}{\Lambda} \right). \quad (2.6)$$

### 2.3 Toric geometry of ALE spaces

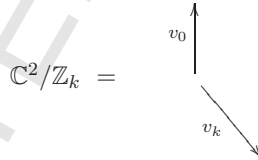
A natural class of complex surfaces  $X$  on which these considerations may be extended consist of orbifolds of  $\mathbb{C}^2$  and their resolutions. We begin by describing ALE spaces of type  $A_{k-1}$  regarded as toric varieties, following [51, 30, 26]; in this paper all cones are understood to be strictly convex rational polyhedral cones in a real vector space. For any non-negative integer  $i$ , define the lattice vector in  $L$  by

$$v_i = i e_1 - (i - 1) e_2.$$

Given an integer  $k \geq 2$ , let the cyclic group  $\mathbb{Z}_k$  of order  $k$  act on  $\mathbb{C}^2$  by

$$\zeta \triangleright (z_1, z_2) = (\omega z_1, \omega^{-1} z_2), \quad (2.7)$$

where  $\zeta$  is the generator of  $\mathbb{Z}_k$  with  $\zeta^k = 1$  and  $\omega = e^{2\pi i/k}$  is a primitive  $k$ -th root of unity. The  $\mathbb{Z}_k$ -action commutes with the  $\mathbb{T}^2$ -action on  $\mathbb{C}^2$ , so  $\mathbb{C}^2/\mathbb{Z}_k$  is an affine toric variety defined by the fan consisting of the single two-dimensional simplicial cone  $\mathbb{R}_{\geq 0}v_0 + \mathbb{R}_{\geq 0}v_k$  spanned by the lattice vectors  $v_0$  and  $v_k$ ; it has a unique singular  $\mathbb{T}^2$ -fixed point at the origin of order  $k$  and can be depicted schematically as



The toric variety  $\mathbb{C}^2/\mathbb{Z}_k = \text{Spec}(\mathbb{C}[z_1, z_2]^{\mathbb{Z}_k})$  corresponding to this fan is dual to the coordinate subalgebra of invariant polynomials  $\mathbb{C}[z_1, z_2]^{\mathbb{Z}_k} = \mathbb{C}[z_1^k, z_2^k, z_1 z_2]$ ; hence the singular orbit space  $\mathbb{C}^2/\mathbb{Z}_k$  can also be regarded as the subvariety of  $\mathbb{C}^3$  cut out by the equation

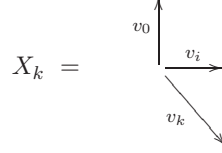
$$x y - z^k = 0. \quad (2.8)$$

Let  $X_k$  be the toric resolution of  $\mathbb{C}^2/\mathbb{Z}_k$ , defined by a simplicial fan  $\Sigma_k$  in the real vector space  $L_{\mathbb{R}} = L \otimes_{\mathbb{Z}} \mathbb{R}$ . Let  $\Sigma_k(n)$ ,  $n = 0, 1, 2$ , be the set of  $n$ -dimensional cones in  $\Sigma_k$ , so that

$$\Sigma_k(0) = \{\{0\}\},$$

$$\begin{aligned}\Sigma_k(1) &= \{\mathbb{R}_{\geq 0}v_i\}_{i=0,1,\dots,k}, \\ \Sigma_k(2) &= \{\mathbb{R}_{\geq 0}v_{i-1} + \mathbb{R}_{\geq 0}v_i\}_{i=1,\dots,k}.\end{aligned}$$

This fan is a subdivision of the fan describing the orbit space  $\mathbb{C}^2/\mathbb{Z}_k$  which can be depicted schematically as



An *ALE space* is a smooth Riemannian four-manifold which is diffeomorphic to  $X_k$  and carries a Kähler metric that is asymptotically locally Euclidean (ALE), i.e. that approximates the standard flat metric on the orbit space  $\mathbb{C}^2/\mathbb{Z}_k$  “at infinity”. It can be realised by adding polynomials in  $\mathbb{C}[x, y, z]$  to (2.8) of degree  $< k$ .

A natural normal toric compactification of  $\mathbb{C}^2/\mathbb{Z}_k$  is the global quotient  $\mathbb{P}^2/\mathbb{Z}_k$ , where  $\mathbb{Z}_k$  acts on the projective plane  $\mathbb{P}^2$  by

$$\zeta \triangleright [w_0, w_1, w_2] = [w_0, \omega w_1, \omega^{-1} w_2], \quad (2.9)$$

and the torus action is given by  $t \triangleright [w_0, w_1, w_2] = [w_0, t_1 w_1, t_2 w_2]$ ; it is a projective toric surface with finite quotient singularities. Let  $\ell_\infty = \{[0, w_1, w_2] \in \mathbb{P}^2\} \cong \mathbb{P}^1$  be a smooth  $\mathbb{T}^2$ -invariant divisor in the projective plane with self-intersection number  $\ell_\infty \cdot \ell_\infty = 1$ . There is a disjoint union

$$\mathbb{P}^2/\mathbb{Z}_k = (\mathbb{C}^2/\mathbb{Z}_k) \cup (\ell_\infty/\mathbb{Z}_k).$$

Note that for  $k = 2$ , the  $\mathbb{Z}_2$ -action on  $\ell_\infty$  is trivial as  $[-w_1, -w_2] = [w_1, w_2]$ ; in general the two fixed points of  $\ell_\infty$  are orbifold points and hence it is a “football”. Then the orbit space  $\mathbb{P}^2/\mathbb{Z}_k$  is the coarse moduli space underlying the global quotient stack  $[\mathbb{P}^2/\mathbb{Z}_k]$  which is the compact toric orbifold defined by the stacky fan [20]  $\Sigma'_k = (L, \Sigma'_k, \beta'_k)$ , where  $\Sigma'_k \subset L_\mathbb{R}$  is the simplicial fan with

$$\begin{aligned}\Sigma'_k(0) &= \{\{0\}\}, \\ \Sigma'_k(1) &= \{\mathbb{R}_{\geq 0}v_0, \mathbb{R}_{\geq 0}v_k, \mathbb{R}_{\geq 0}(-v_0 - v_k)\}, \\ \Sigma'_k(2) &= \{\mathbb{R}_{\geq 0}v_0 + \mathbb{R}_{\geq 0}v_k, \mathbb{R}_{\geq 0}v_0 + \mathbb{R}_{\geq 0}(-v_0 - v_k), \mathbb{R}_{\geq 0}v_k + \mathbb{R}_{\geq 0}(-v_0 - v_k)\},\end{aligned}$$

and  $\beta'_k : \mathbb{Z}b'_1 \oplus \mathbb{Z}b'_2 \oplus \mathbb{Z}b'_3 \rightarrow L$  is the map

$$\beta'_k : \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 0 & k & -k \\ 1 & k-1 & -k \end{pmatrix}} \mathbb{Z}^2$$

determined by the minimal lattice points  $b'_1, b'_2, b'_3$  generating the one-cones  $\Sigma'_k(1)$ ; here and in the following we shall usually implicitly assume that  $k$  is odd for simplicity (see [26] for the general case), although many of our conclusions hold more generally. Note that for  $k = 2$ , the divisor  $[\ell_\infty/\mathbb{Z}_2] \cong \mathbb{P}^1 \times B\mathbb{Z}_2$  corresponding to the vector  $b'_3$  is a trivial  $\mathbb{Z}_2$ -gerbe (the quotient stack of  $\mathbb{P}^1$  by the trivial  $\mathbb{Z}_2$ -action), where  $B\mathbb{Z}_2$  is the quotient groupoid  $[\text{Spec}(\mathbb{C})/\mathbb{Z}_2]$ .

Minimal resolution of the singularity at the origin  $[1, 0, 0]$  gives a stacky toric compactification

$$[\overline{X}_k] = [X_k \cup (\ell_\infty/\mathbb{Z}_k)]$$

of  $X_k$ , which is defined by a stacky fan  $\bar{\Sigma}_k = (L, \bar{\Sigma}_k, \bar{\beta}_k)$ , where  $\bar{\Sigma}_k \subset L_{\mathbb{R}}$  is the simplicial fan with

$$\begin{aligned}\bar{\Sigma}_k(0) &= \{\{0\}\}, \\ \bar{\Sigma}_k(1) &= \{\mathbb{R}_{\geq 0}v_i\}_{i=0,1,\dots,k} \cup \{\mathbb{R}_{\geq 0}(-v_0 - v_k)\}, \\ \bar{\Sigma}_k(2) &= \{\mathbb{R}_{\geq 0}v_{i-1} + \mathbb{R}_{\geq 0}v_i\}_{i=1,\dots,k} \\ &\quad \cup \{\mathbb{R}_{\geq 0}v_0 + \mathbb{R}_{\geq 0}(-v_0 - v_k), \mathbb{R}_{\geq 0}v_k + \mathbb{R}_{\geq 0}(-v_0 - v_k)\},\end{aligned}$$

and  $\bar{\beta}_k : \mathbb{Z}\bar{b}_1 \oplus \dots \oplus \mathbb{Z}\bar{b}_{k+2} \rightarrow L$  is the map

$$\bar{\beta}_k : \mathbb{Z}^{k+2} \xrightarrow{\begin{pmatrix} 0 & 1 & 2 & \dots & k & -k \\ 1 & 0 & 1 & \dots & k-1 & -k \end{pmatrix}} \mathbb{Z}^2$$

defined by the minimal lattice points  $\bar{b}_1, \dots, \bar{b}_{k+2}$  generating the one-cones  $\bar{\Sigma}_k(1)$ . The coarse moduli space  $\overline{X}_k$  of this toric orbifold is the simplicial toric variety defined by the fan  $\bar{\Sigma}_k$ ; in particular for  $k=2$ ,  $\overline{X}_2$  is the second Hirzebruch surface  $\mathbb{F}_2$ .

Let  $\Gamma$  be the toric graph with vertex set  $V(\Gamma) = \{v_1, \dots, v_k\}$  given by vertices of two-cones in  $\Sigma_k(2)$ , which is in bijective correspondence with the set of smooth torus fixed points  $p_i = p_{v_i}$  in  $X_k$ , i.e.  $X_k^{\mathbb{T}^2} = \{p_1, \dots, p_k\}$ . The tangent weights of the toric action on the  $\mathbb{T}^2$ -fixed point  $p_i$  are

$$w_1^i = -(i-2)\epsilon_1 - (i-1)\epsilon_2 \quad \text{and} \quad w_2^i = (i-1)\epsilon_1 - i\epsilon_2$$

for  $i = 1, \dots, k$ . Let  $U[\sigma_i]$  be the affine toric variety generated by the smooth two-cones  $\sigma_i := \mathbb{R}_{\geq 0}v_{i-1} + \mathbb{R}_{\geq 0}v_i$  for  $i = 1, \dots, k$ . Then

$$X_k = \bigcup_{i=1}^k U[\sigma_i].$$

Let  $E(\Gamma) = \{e_1, \dots, e_{k-1}\}$  be the set of edges in the graph  $\Gamma$ , where  $e_i$  is the edge connecting vertices  $v_i$  and  $v_{i+1}$ , and let  $\ell_i = \ell_{e_i}$  be the corresponding exceptional divisors of the minimal resolution  $X_k \rightarrow \mathbb{C}^2/\mathbb{Z}_k$ . Then  $\ell_i$  is a  $\mathbb{T}^2$ -invariant projective line connecting the  $\mathbb{T}^2$ -fixed points  $p_i$  and  $p_{i+1}$ . For each  $i = 1, \dots, k-1$  and  $j = 1, \dots, k$ , there is a unique lattice vector  $m_i^j \in L^*$  such that

$$\ell_i|_{U[\sigma_j]} = - \sum_{l=1}^{k-1} \langle m_i^j, v_l \rangle \ell_l|_{U[\sigma_j]};$$

it is given explicitly by

$$m_i^j = \begin{cases} (i-2)e_1^* + (i-1)e_2^*, & j = i, \\ -ie_1^* - (i+1)e_2^*, & j = i+1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence the stalk of the associated line bundle  $\mathcal{O}_{X_k}(\ell_i)$  has weights at  $p_j$  given by

$$w_{\ell_i}^{p_j} = \begin{cases} -(i-2)\epsilon_1 - (i-1)\epsilon_2, & j = i, \\ i\epsilon_1 + (i+1)\epsilon_2, & j = i+1, \\ 0, & \text{otherwise} \end{cases}$$

for  $i = 1, \dots, k-1$  and  $j = 1, \dots, k$ . The intersection form  $C$  of the exceptional divisors coincides with minus the  $(k-1) \times (k-1)$  Cartan matrix

$$C = (\ell_i \cdot \ell_j) = \begin{pmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & \dots & 0 \\ 0 & 1 & -2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -2 \end{pmatrix}$$

of the Dynkin diagram for the  $A_{k-1}$  Lie algebra. Since  $X_k$  is non-compact, the intersection form is not unimodular and the inverse  $C^{-1}$  has generically rational-valued matrix elements given by [54]

$$(C^{-1})^{ij} = \frac{i \cdot j}{k} - \min(i, j)$$

for  $i, j = 1, \dots, k-1$ .

Any compactly supported divisor  $D \in H_c^2(X_k, \mathbb{Z})$  is a linear combination

$$D = \sum_{i=1}^{k-1} m_i \ell_i \quad \text{with } m_i \in \mathbb{Z},$$

so that the corresponding weights are

$$w_D^{p_j} = \sum_{i=1}^{k-1} m_i w_{\ell_i}^{p_j} = ((j-1)\epsilon_1 + j\epsilon_2) m_{j-1} - ((j-2)\epsilon_1 + (j-1)\epsilon_2) m_j$$

for  $j = 1, \dots, k$ , where  $m_0 = m_k := 0$ . However, to properly account for flat line bundles on  $X_k$  with non-trivial holonomy at infinity, we define a dual generating set, with respect to the intersection pairing linearly extended to non-compact divisors, by [30]

$$\varepsilon^i = \sum_{j=1}^{k-1} (C^{-1})^{ij} \ell_j.$$

Then  $\varepsilon^i$ ,  $i = 1, \dots, k-1$ , extend the pair of non-compact torically invariant divisors of  $X_k$  to an integral generating set for the Picard group of line bundles

$$\text{Pic}(X_k) = H^2(X_k, \mathbb{Z}) \cong \mathbb{Z}^k,$$

with intersection product  $\varepsilon^i \cdot \varepsilon^j = (C^{-1})^{ij}$ ; this set corresponds to the basis of tautological line bundles  $\mathcal{R}_1, \dots, \mathcal{R}_{k-1}$  constructed by Kronheimer and Nakajima in [67] with  $\int_{X_k} c_1(\mathcal{R}_i) \wedge c_1(\mathcal{R}_j) = (C^{-1})^{ij}$ . We can then parameterize the class of a divisor  $D = D_{\vec{u}}$  as

$$D_{\vec{u}} = \sum_{i=1}^{k-1} u_i \varepsilon^i$$

with  $\vec{u} = (u_1, \dots, u_{k-1}) \in \mathbb{Z}^{k-1}$ . The corresponding (fractional) weights are

$$w_{D_{\vec{u}}}^{p_i} = \sum_{j,l=1}^{k-1} u_j (C^{-1})^{jl} w_{\ell_l}^{p_i} = -\frac{(k-1)\epsilon_1 + k\epsilon_2}{k} \sum_{j=1}^{i-1} j u_j - \frac{\epsilon_1}{k} \sum_{j=i}^{k-1} (k-j) u_j$$

for  $i = 1, \dots, k$ .

## 2.4 Quiver varieties

We will now consider the extensions of the moduli spaces  $\mathfrak{M}_{N,n}(\mathbb{C}^2)$  to the more general family of cyclic Nakajima quiver varieties. As in the case of gauge theories on  $\mathbb{R}^4$ ,  $U(N)$  instantons on  $X_k$  are asymptotically flat, i.e. they approach flat connections at infinity. However, while flat connections at infinity on  $\mathbb{R}^4$  are necessarily trivial, on  $X_k$  they are not: At infinity  $X_k$  looks topologically like the lens space  $S^3/\mathbb{Z}_k$ , so a flat connection can have non-trivial holonomy which is parameterized by a representation  $\rho$  of the fundamental group  $\pi_1(S^3/\mathbb{Z}_k) \cong \mathbb{Z}_k$ . On the toric compactification  $\overline{X}_k$  they correspond to flat bundles over the compactification divisor  $\ell_\infty/\mathbb{Z}_k$ . For this, let  $\rho : \mathbb{Z}_k \rightarrow GL(N, \mathbb{C})$  be a representation of the cyclic group  $\mathbb{Z}_k$ . Then the flat bundle  $E_\infty \rightarrow \ell_\infty/\mathbb{Z}_k$  of rank  $N$  associated to  $\rho$  is

$$E_\infty := \mathbb{P}^1 \times_{\mathbb{Z}_k} \mathbb{C}^N \longrightarrow \mathbb{P}^1/\mathbb{Z}_k ,$$

where the generator  $\zeta$  of  $\mathbb{Z}_k$  acts on  $\mathbb{P}^1 \times \mathbb{C}^N$  as

$$\zeta \triangleright ([w_1, w_2], v) = ([\omega w_1, \omega^{-1} w_2], \rho(\zeta)v) .$$

Let  $\rho_i : \mathbb{Z}_k \rightarrow U(1)$  denote the irreducible one-dimensional representation of  $\mathbb{Z}_k$  with weight  $i$  for  $i = 0, 1, \dots, k-1$ . By the McKay correspondence, the corresponding flat line bundles  $E_\infty$  are precisely the restrictions of the tautological line bundles  $\mathcal{R}_i$  at infinity, where  $\mathcal{R}_0 := \mathcal{O}_{X_k}$ .

In [67], Kronheimer and Nakajima construct moduli spaces  $\mathfrak{M}(\vec{v}, \vec{w})$  of  $U(N)$  instantons on  $X_k$  with the vectors  $\vec{v}, \vec{w} \in \mathbb{Z}_{\geq 0}^k$  parameterizing the Chern classes  $\text{ch}(\mathcal{E})$  and the holonomies  $\rho = \rho_{\vec{w}} := \bigoplus_{i=0}^{k-1} w_i \rho_i$ . For this, we recall the parameterization of the moduli spaces  $\mathfrak{M}_{N,n}(\mathbb{C}^2)$  by linear algebraic ADHM data which neatly captures the instanton quantum mechanics of  $n$  D0-branes inside  $N$  D4-branes on  $\mathbb{C}^2$ : One has  $\mathfrak{M}_{N,n}(\mathbb{C}^2) = \mu^{-1}(0)^\xi/GL(n, \mathbb{C})$ , where

$$\mu = [b_1, b_2] + i j . \quad (2.10)$$

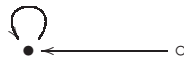
Here  $(b_1, b_2) \in \text{End}_{\mathbb{C}}(V) \otimes Q$ ,  $i \in \text{Hom}_{\mathbb{C}}(W, V)$  and  $j \in \text{Hom}_{\mathbb{C}}(V, W) \otimes \bigwedge^2 Q$ , where  $V$  and  $W$  are complex vector spaces of dimensions  $n$  and  $N$ , respectively, while  $Q \cong \mathbb{C}^2$  is the fundamental representation of  $\mathbb{T}^2$  with weight  $(1, 1)$  and  $W \cong \mathbb{C}^N$  the fundamental representation of the torus  $(\mathbb{C}^\times)^N$  with weight  $(1, \dots, 1)$ . The group  $GL(n, \mathbb{C})$  acts by basis change of  $V \cong \mathbb{C}^n$ , and the superscript  $\xi$  indicates that the GIT quotient is restricted to *stable* matrices  $(b_1, b_2, i, j)$ , i.e. the image of  $i$  generates  $V$  under the action of  $b_1, b_2$ . This enables one to interpret  $\mathfrak{M}_{N,n}(\mathbb{C}^2)$  as the moduli variety of stable linear representations of the ADHM quiver

$$\begin{array}{c} \begin{array}{c} \left. \begin{array}{c} b_1 \\ \bullet \\ b_2 \end{array} \right\} \\ V \end{array} \begin{array}{c} \xrightarrow{j} \\ \xleftarrow{i} \end{array} \begin{array}{c} \circ \\ W \end{array} \end{array} \quad (2.11)$$

with relations (2.10), which can be regarded as originating in the following way: The Jordan quiver

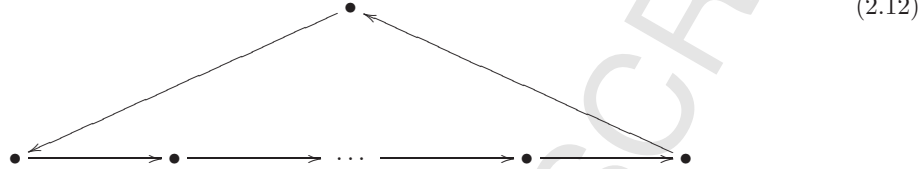


has as underlying graph the (formal) affine extended Dynkin diagram of type  $\hat{A}_0$ . The corresponding framed quiver (obtained by adding a node and arrow to each node) is



and its double (obtained by adding an arrow in the opposite direction to each arrow) is precisely the ADHM quiver (2.11). Alternatively,  $\mathfrak{M}_{N,n}(\mathbb{C}^2)$  can be regarded geometrically as a symplectic reduction of the cotangent bundle over the moduli space of representations of the framed Jordan quiver; passing to the cotangent bundle has the effect of doubling the quiver.

One can now take  $V$  and  $W$  to be  $\mathbb{Z}_k$ -modules; then  $\vec{v}$  and  $\vec{w}$  are dimension vectors whose components give the multiplicities of their decompositions into irreducible  $\mathbb{Z}_k$ -modules. By decomposing the linear maps  $(b_1, b_2, i, j)$  of the ADHM construction as equivariant maps  $(b_1, b_2) \in \text{End}_{\mathbb{Z}_k}(V) \otimes Q$ ,  $i \in \text{Hom}_{\mathbb{Z}_k}(W, V)$  and  $j \in \text{Hom}_{\mathbb{Z}_k}(V, W) \otimes \wedge^2 Q$ , one can define a moduli variety  $\mathfrak{M}(\vec{v}, \vec{w})$  of stable linear representations of the double of the framed quiver corresponding to the cyclic quiver



whose underlying graph is the affine extended Dynkin diagram of type  $\hat{A}_{k-1}$ , together with relations provided by equivariant decomposition of (2.10). More generally, one can define the Nakajima quiver varieties  $\mathfrak{M}_\xi(\vec{v}, \vec{w})$  of type  $\hat{A}_{k-1}$  which depend on a suitable stability parameter  $\xi \in H^2(X_k, \mathbb{R}) = \mathbb{R}^k$  [78]. They are smooth quasi-projective varieties of dimension

$$\dim_{\mathbb{C}} \mathfrak{M}_\xi(\vec{v}, \vec{w}) = 2\vec{w} \cdot \vec{v} - \vec{v} \cdot \hat{C}\vec{v},$$

where

$$\hat{C} = \begin{pmatrix} 2 & -1 & 0 & \dots & -1 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & 2 \end{pmatrix}$$

is the Cartan matrix of the extended Dynkin diagram of type  $\hat{A}_{k-1}$ . The space of parameters  $\xi$  such that stable points coincide with semistable points (in the representation variety) has a subdivision into chambers, whereby the quiver varieties  $\mathfrak{M}_\xi(\vec{v}, \vec{w})$  with parameters in the same chamber are isomorphic (as complex algebraic varieties) while those with parameters in distinct chambers are only diffeomorphic. In this paper we will focus on two distinguished chambers.

There is a chamber  $C_0$  such that, for  $\xi_0 \in C_0$ , the quiver variety  $\mathfrak{M}_{\xi_0}(\vec{v}, \vec{w})$  parameterizes  $\mathbb{Z}_k$ -equivariant torsion-free sheaves  $\mathcal{E}$  on  $\mathbb{P}^2$  with a  $\mathbb{Z}_k$ -invariant framing isomorphism  $\phi_{\mathcal{E}} : \mathcal{E}|_{\ell_\infty} \rightarrow \mathcal{O}_{\ell_\infty} \otimes \rho_{\vec{w}}$  and  $H^1(\mathbb{P}^2, \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(\ell_\infty)) \cong V$ . The  $\mathbb{Z}_k$ -action on  $\mathbb{P}^2$  lifts to a natural  $\mathbb{Z}_k$ -action on the moduli space  $\mathfrak{M}_{N,n}(\mathbb{C}^2)$  and one can compute equivariant characters by taking the  $\mathbb{Z}_k$ -invariant part of the pertinent  $\mathbb{T}^2$ -modules on  $\mathfrak{M}_{N,n}(\mathbb{C}^2)$ ; in fact there is a decomposition of the  $\mathbb{Z}_k$ -fixed point set

$$\mathfrak{M}_{N,n}(\mathbb{C}^2)^{\mathbb{Z}_k} = \bigsqcup_{|\vec{v}|=n} \mathfrak{M}_{\xi_0}(\vec{v}, \vec{w})$$

with  $|\vec{v}| := \sum_{i=0}^{k-1} v_i$ , for fixed  $\vec{w} \in \mathbb{Z}_{\geq 0}^k$  with  $|\vec{w}| = N$ . This enables the computation of partition functions, regarded as integrals over the moduli spaces  $\mathfrak{M}_{\xi_0}(\vec{v}, \vec{w})$ , for  $N = 2$  gauge theory on the resolution of the Kleinian singularity  $\mathbb{C}^2/\mathbb{Z}_k$  provided by the quotient stack  $[\mathbb{C}^2/\mathbb{Z}_k]$  [46, 47]. As an explicit example, in the following we treat the case of  $U(1)$  gauge theory in detail. The  $\mathbb{Z}_k$ -action on  $\mathbb{C}^2$  given by (2.7) endows  $\mathbb{C}[z_1, z_2]$  with a canonical  $\mathbb{Z}_k$ -module structure which lifts to a  $\mathbb{Z}_k$ -action on



the Hilbert schemes  $\text{Hilb}^n(\mathbb{C}^2)$ . We set  $\vec{w} = \vec{w}_0 := (1, 0, \dots, 0)$  and fix  $\vec{v} = (v_0, v_1, \dots, v_{k-1}) \in \mathbb{Z}_{\geq 0}^k$ . Then  $\mathfrak{M}_{\xi_0}(\vec{v}, \vec{w}_0) \cong \text{Hilb}^{|\vec{v}|}(\mathbb{C}^2)^{\mathbb{Z}_k, \vec{v}}$  is the moduli space parameterizing  $\mathbb{Z}_k$ -invariant ideals  $I$  of codimension  $|\vec{v}|$  in the polynomial ring  $\mathbb{C}[z_1, z_2]$  such that  $\mathbb{C}[z_1, z_2]/I \cong \bigoplus_{i=0}^{k-1} v_i \rho_i$ . It is a smooth quasi-projective variety of dimension

$$\dim_{\mathbb{C}} \text{Hilb}^{|\vec{v}|}(\mathbb{C}^2)^{\mathbb{Z}_k, \vec{v}} = 2v_0 - \vec{v} \cdot \widehat{C}\vec{v}.$$

There is another distinguished chamber  $C_{\infty}$  such that, for  $\xi_{\infty} \in C_{\infty}$  and for  $\vec{v} = \vec{\delta} = (1, \dots, 1)$ , the moduli space  $\mathfrak{M}_{\xi_{\infty}}(\vec{\delta}, \vec{w}_0)$  is the  $\mathbb{Z}_k$ -Hilbert scheme of the plane  $\mathbb{C}^2$  with  $\widehat{C}\vec{\delta} = \vec{0}$ ,  $|\vec{\delta}| = k$  and  $\text{Hilb}^{\mathbb{Z}_k}(\mathbb{C}^2) := \text{Hilb}^{|\vec{\delta}|}(\mathbb{C}^2)^{\mathbb{Z}_k, \vec{\delta}} \cong X_k$  under the Hilbert–Chow morphism  $\text{Hilb}^{\mathbb{Z}_k}(\mathbb{C}^2) \rightarrow \mathbb{C}^2/\mathbb{Z}_k$  (cf. [47, §3.1.2]); in this case  $\mathbb{C}[z_1, z_2]/I$  is isomorphic to the regular representation of the cyclic group  $\mathbb{Z}_k$ . This is the complex algebraic version of Kronheimer’s construction of ALE spaces [66]. Since the quiver varieties  $\mathfrak{M}_{\xi_0}(\vec{v}, \vec{w})$  and  $\mathfrak{M}_{\xi_{\infty}}(\vec{v}, \vec{w})$  are *not* isomorphic, they have distinct universal bundles. As it is difficult (for us) to work directly over the Nakajima quiver varieties, one needs to independently develop a means for studying gauge theories with instanton moduli associated to the chamber  $C_{\infty}$ . Such a theory was developed by [26] in the framework of framed sheaves on a suitable orbifold compactification  $\mathcal{X}_k = X_k \cup \mathcal{D}_{\infty}$ , which is a smooth projective toric orbifold. The compactification divisor  $\mathcal{D}_{\infty}$  is a  $\mathbb{Z}_k$ -gerbe over a football which as a toric Deligne–Mumford stack has a presentation as a global quotient stack

$$\mathcal{D}_{\infty} \cong \left[ \frac{\mathbb{C}^2 \setminus \{0\}}{\mathbb{C}^{\times} \times \mathbb{Z}_k} \right]$$

with Deligne–Mumford torus  $\mathbb{C}^{\times} \times B\mathbb{Z}_k$ . Hence its Picard group is given by  $\text{Pic}(\mathcal{D}_{\infty}) \cong \mathbb{Z} \oplus \mathbb{Z}_k$  and we denote the respective generators of the two factors by  $\mathcal{L}_1, \mathcal{L}_2$ . The fundamental group of the underlying topological stack is given by  $\pi_1(\mathcal{D}_{\infty}) \cong \mathbb{Z}_k$ , and each line bundle  $\mathcal{O}_{\mathcal{D}_{\infty}}(i) = \mathcal{L}_2^{\otimes i}$  can be endowed with a unitary flat connection associated to the representation  $\rho_i : \mathbb{Z}_k \rightarrow U(1)$  [42]. The Picard group  $\text{Pic}(\mathcal{X}_k)$  is generated by the line bundles  $\mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_{\infty})$  and  $\mathcal{R}_1, \dots, \mathcal{R}_{k-1}$ , where the restrictions of  $\mathcal{R}_i$  to  $X_k$  coincide with the tautological line bundles  $\mathcal{R}_i$  and to  $\mathcal{D}_{\infty}$  with  $\mathcal{O}_{\mathcal{D}_{\infty}}(i)$ .

For fixed  $\vec{w} = (w_0, w_1, \dots, w_{k-1}) \in \mathbb{Z}_{\geq 0}^k$ , by using the general theory of framed sheaves on projective stacks developed by [23] one can construct a fine moduli space  $\mathfrak{M}_{\vec{u}, \Delta, \vec{w}}(X_k)$  parameterizing torsion-free sheaves  $\mathcal{E}$  on  $\mathcal{X}_k$  with a framing isomorphism  $\phi_{\mathcal{E}} : \mathcal{E}|_{\mathcal{D}_{\infty}} \rightarrow \bigoplus_{i=0}^{k-1} \mathcal{O}_{\mathcal{D}_{\infty}}(i)^{\oplus w_i}$  of rank  $N = \sum_{i=0}^{k-1} w_i$ , first Chern class  $c_1(\mathcal{E}) = \sum_{i=1}^{k-1} u_i c_1(\mathcal{R}_i)$ , and discriminant  $\Delta(\mathcal{E}) = \Delta$  where

$$\Delta(\mathcal{E}) = \int_{\mathcal{X}_k} \left( c_2(\mathcal{E}) - \frac{N-1}{N} c_1(\mathcal{E})^2 \right).$$

The framing condition restricts the Chern classes to  $\mathcal{U}_{\vec{w}}$  which is the set of integer vectors  $\vec{u} \in \mathbb{Z}^{k-1}$  that correspond to the sum of the weight  $\sum_{i=1}^{k-1} w_i \omega_i$  and an element  $\gamma_{\vec{u}} \in \mathcal{Q}$  of the  $A_{k-1}$  root lattice; here  $\omega_1, \dots, \omega_{k-1}$  are the fundamental weights of type  $A_{k-1}$ . The moduli space  $\mathfrak{M}_{\vec{u}, \Delta, \vec{w}}(X_k)$  is a smooth quasi-projective variety of dimension

$$\dim_{\mathbb{C}} \mathfrak{M}_{\vec{u}, \Delta, \vec{w}}(X_k) = 2r\Delta - \frac{1}{2} \sum_{j=1}^{k-1} (C^{-1})^{jj} \vec{w} \cdot \vec{w}(j),$$

where  $\vec{w}(j) := (w_j, \dots, w_{k-1}, w_0, w_1, \dots, w_{j-1})$ , and it contains the moduli space of  $U(N)$  instantons on  $X_k$  of first Chern class  $\sum_i u_i c_1(\mathcal{R}_i)$  and holonomy at infinity  $\rho = \bigoplus_j w_j \rho_j$  [42]. One furthermore has [26, 42]

**Theorem 2.13** *There is a birational morphism*

$$\mathfrak{M}_{\vec{u}, \Delta, \vec{w}}(X_k) \longrightarrow \mathfrak{M}_{\xi_{\infty}}(\vec{v}, \vec{w})$$

for some  $\vec{v} \in \mathbb{Z}_{\geq 0}^k$ .

In the rank one case  $N = 1$ , this morphism is an isomorphism: In this instance  $\mathfrak{M}_{\vec{u}, n, \vec{w}}(X_k) \cong \text{Hilb}^n(X_k)$  is the Hilbert scheme of  $n$  points on  $X_k$  for all  $\vec{u}$  and  $\vec{w}$  [26], which is a Nakajima quiver variety by [68].

### 2.5 $N = 2$ gauge theory on $X_k$

Let us first present the computation of the instanton partition function corresponding to the chamber  $\mathbb{C}_\infty$ . For pure  $N = 2$  gauge theory on  $X_k$ , it is defined by a weighted sum over topological sectors  $\vec{u}$  of fractional instantons with chemical potentials  $\vec{\xi} = (\xi_1, \dots, \xi_{k-1})$  as

$$\mathcal{Z}_{X_k}^{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; \mathbf{q}, \vec{\xi})_{\vec{w}} = \sum_{\vec{u} \in \mathcal{U}_{\vec{w}}} \vec{\xi}^{\vec{u}} \sum_{\Delta \in \frac{1}{2Nk} \mathbb{Z}} \mathbf{q}^{\Delta + \frac{1}{2N} \vec{u} \cdot C^{-1} \vec{u}} \int_{\mathfrak{M}_{\vec{u}, \Delta, \vec{w}}(X_k)} 1,$$

where  $\vec{\xi}^{\vec{u}} = \prod_{i=1}^{k-1} \xi_i^{u_i}$ . The torus fixed points  $\mathfrak{M}_{\vec{u}, \Delta, \vec{w}}(X_k)^{\tilde{\Gamma}}$  are parameterized by vectors of Young diagrams  $\vec{Y} = (Y_1, \dots, Y_N)$ , with  $Y_l = \{Y_l^i\}_{i=1, \dots, k}$ , and vectors of integers  $\vec{u} = (u_1, \dots, u_N)$  such that  $\vec{u} = \sum_{l=1}^N \vec{u}_l$  with  $\vec{u}_l$  corresponding to  $\gamma_{\vec{u}_l} + \omega_i$  for  $i = 1, \dots, k-1$  and  $\sum_{j=0}^{i-1} w_j < l \leq \sum_{j=0}^i w_j$ , and

$$\Delta = \sum_{l=1}^N |\vec{Y}_l| + \frac{1}{2} \sum_{l=1}^N \vec{u}_l \cdot C^{-1} \vec{u}_l - \frac{1}{2N} \sum_{l, l'=1}^N \vec{u}_l \cdot C^{-1} \vec{u}_{l'}.$$

The localization theorem then gives the factorization formula

$$\begin{aligned} \mathcal{Z}_{X_k}^{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; \mathbf{q}, \vec{\xi})_{\vec{w}} &= \sum_{\vec{u} \in \mathcal{U}_{\vec{w}}} \vec{\xi}^{\vec{u}} \sum_{\vec{u}} \mathbf{q}^{\frac{1}{2} \sum_{l=1}^N \vec{u}_l \cdot C^{-1} \vec{u}_l} \\ &\times \prod_{l, l'=1}^N \prod_{n=1}^{k-1} \frac{1}{\ell_{\vec{u}_{l'} - \vec{u}_l}^{(n)}(\epsilon_1^{(n)}, \epsilon_2^{(n)}, a_{l'} - a_l)} \prod_{i=1}^k \mathcal{Z}_{\mathbb{C}^2}^{\text{inst}}(\epsilon_1^{(i)}, \epsilon_2^{(i)}, \vec{a}^{(i)}; \mathbf{q}), \end{aligned}$$

which determines the partition function in terms of the fan of  $X_k$  as a product of  $k$  copies of the instanton partition function (2.2) on  $\mathbb{C}^2$ ; it depends on the equivariant parameters of the torus action on the affine toric patches  $U[\sigma_i] \cong \mathbb{C}^2$  of  $X_k$  (which can be read off from the weights of §2.3), and on the leg factors  $\ell_{\vec{u}}^{(n)}$  which are determined by the geometry of the exceptional divisors  $\ell_1, \dots, \ell_{k-1}$  of the resolution  $X_k \rightarrow \mathbb{C}^2/\mathbb{Z}_k$  (whose explicit forms can be found in [26, 24]). This expression generalizes the Nakajima–Yoshioka blowup formulas [82]; it was originally conjectured in [17, 18, 19] and then rigorously proven in [26]. This form of the partition function is expected to nicely capture physical features of  $N = 2$  gauge theories on the ALE space  $X_k$  and their relations to two-dimensional conformal field theory that we discuss in §3.

In the rank one case  $N = 1$ , with  $\vec{w}_j$  the  $j$ -th coordinate vector of  $\mathbb{Z}^k$  for  $j = 0, 1, \dots, k-1$ , the leg factors are unity and the partition function simplifies to [91]

$$\mathcal{Z}_{X_k}^{\text{inst}}(\epsilon_1, \epsilon_2; \mathbf{q}, \vec{\xi})_{\vec{w}_j} = \widehat{\eta}(\mathbf{q})^{k-1} \chi^{\widehat{w}_j}(\mathbf{q}, \vec{\xi}) \exp\left(\frac{\mathbf{q}}{k \epsilon_1 \epsilon_2}\right), \quad (2.14)$$

where

$$\chi^{\widehat{w}_j}(\mathbf{q}, \vec{\xi}) = \frac{1}{\widehat{\eta}(\mathbf{q})^{k-1}} \sum_{\vec{u} \in \mathcal{U}_{\vec{w}_j}} \mathbf{q}^{\frac{1}{2} \vec{u} \cdot C^{-1} \vec{u}} \vec{\xi}^{\vec{u}}$$

is the character of the integrable highest weight representation of the affine Lie algebra  $\widehat{\mathfrak{sl}}_k$  at level one, with highest weight the  $j$ -th fundamental weight  $\widehat{\omega}_j$  of type  $\widehat{A}_{k-1}$  for  $j = 0, 1, \dots, k-1$ , and

$$\widehat{\eta}(q)^{-1} = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

is the character of the Fock space representation of the Heisenberg algebra  $\mathfrak{h}$ , i.e. the Euler function which is the generating function for Young diagrams. This formula generalizes the  $k=1$  expression (2.3); its representation theoretic content will be elucidated further in §3.

## 2.6 $\mathcal{N}=2$ gauge theory on $[\mathbb{C}^2/\mathbb{Z}_k]$

Let us now compare the computation of §2.5 with that corresponding to the chamber  $\mathcal{C}_0$ ; see §A.2 for the relevant combinatorial definitions and properties used below. Consider again the rank one case to begin with. The instanton part of Nekrasov's partition function for the pure  $\mathcal{N}=2$  gauge theory on the quotient stack  $[\mathbb{C}^2/\mathbb{Z}_k]$  is defined by

$$\mathcal{Z}_{[\mathbb{C}^2/\mathbb{Z}_k]}^{\text{inst}}(\epsilon_1, \epsilon_2; \tilde{\mathbf{q}}, \vec{\mathbf{q}}) := \sum_{\vec{v} \in \mathbb{Z}_{\geq 0}^k} \tilde{\mathbf{q}}^{|\vec{v}|} \vec{\mathbf{q}}^{\vec{v}} \int_{\text{Hilb}^{|\vec{v}|}(\mathbb{C}^2)^{\mathbb{Z}_k, \vec{v}}} 1,$$

where  $\vec{\mathbf{q}} = (\mathbf{q}_0, \mathbf{q}_1, \dots, \mathbf{q}_{k-1})$  and  $\vec{\mathbf{q}}^{\vec{v}} := \prod_{i=0}^{k-1} \mathbf{q}_i^{v_i}$ . The action of the torus  $\mathbb{T}^2 = \mathbb{C}^\times \times \mathbb{C}^\times$  on  $\text{Hilb}^{|\vec{v}|}(\mathbb{C}^2)$  restricts to a torus action on  $\text{Hilb}^{|\vec{v}|}(\mathbb{C}^2)^{\mathbb{Z}_k, \vec{v}}$ . As described in [47, §3.2], the torus fixed points of  $\text{Hilb}^{|\vec{v}|}(\mathbb{C}^2)^{\mathbb{Z}_k, \vec{v}}$  correspond to  $k$ -coloured Young diagrams  $Y$  with  $|Y| = |\vec{v}|$  such that  $\nu_i(Y) = v_i$  for  $i = 0, 1, \dots, k-1$ . Since all nodes on the line  $b = a - \mu k - i$  for  $\mu \in \mathbb{Z}$  are coloured by the same colour  $i$ , we have

$$v_i = \sum_{\mu \in \mathbb{Z}} N_{\mu k + i}(Y).$$

Since an  $r$ -hook for  $r = n k$  has  $n$   $i$ -nodes for each  $i = 0, 1, \dots, k-1$  and  $c_h(Y) = v_{h-\frac{1}{2}} - v_{h+\frac{1}{2}}$ , all  $k$ -coloured Young diagrams  $Y$  corresponding to dimension vectors  $\vec{v} = n \vec{\delta}$  have  $k$ -core  $\vec{c}(Y) = \vec{0}$ , i.e.  $\underline{Y} = \emptyset$ .

By the localization theorem we have

$$\int_{\text{Hilb}^{|\vec{v}|}(\mathbb{C}^2)^{\mathbb{Z}_k, \vec{v}}} 1 = \sum_{\nu_i(Y)=v_i} \prod_{\substack{s \in Y \\ h(s) \equiv 0 \pmod k}} \frac{1}{((L(s)+1)\epsilon_1 - A(s)\epsilon_2)(-L(s)\epsilon_1 + (A(s)+1)\epsilon_2)}.$$

By decomposing Young diagrams  $Y$  into their  $k$ -cores and  $k$ -quotients  $(\vec{c}(Y), \vec{q}(Y))$ , and repeating the same argument as in [47, §4.2.1], we then find the factorization formula

$$\mathcal{Z}_{[\mathbb{C}^2/\mathbb{Z}_k]}^{\text{inst}}(\epsilon_1, \epsilon_2; \tilde{\mathbf{q}}, \vec{\mathbf{q}}) = \mathcal{Z}_{[\mathbb{C}^2/\mathbb{Z}_k]}(\tilde{\mathbf{q}}, \vec{\mathbf{q}})^{\text{core}} \mathcal{Z}_{[\mathbb{C}^2/\mathbb{Z}_k]}(\epsilon_1, \epsilon_2; \tilde{\mathbf{q}}, \vec{\mathbf{q}})^{\text{quot}},$$

where

$$\mathcal{Z}_{[\mathbb{C}^2/\mathbb{Z}_k]}(\tilde{\mathbf{q}}, \vec{\mathbf{q}})^{\text{core}} := \sum_{\vec{q}(Y)=\vec{0}} \tilde{\mathbf{q}}^{|Y|} \prod_{i=0}^{k-1} \mathbf{q}_i^{\sum_{\mu \in \mathbb{Z}} N_{\mu k + i}(Y)}$$

is the contribution from  $k$ -cores (Young diagrams with no  $k$ -hooks), while

$$\mathcal{Z}_{[\mathbb{C}^2/\mathbb{Z}_k]}(\epsilon_1, \epsilon_2; \tilde{\mathbf{q}}, \vec{\mathbf{q}})^{\text{quot}} := \sum_{v_0=v_1=\dots=v_{k-1}} \tilde{\mathbf{q}}^{|\vec{v}|} \vec{\mathbf{q}}^{\vec{v}}$$

$$\times \sum_{\nu_i(Y)=v_i} \prod_{\substack{s \in Y \\ h(s) \equiv 0 \pmod k}} \frac{1}{((L(s)+1)\epsilon_1 - A(s)\epsilon_2)(-L(s)\epsilon_1 + (A(s)+1)\epsilon_2)}$$

is the contribution from  $k$ -quotients ( $\vec{c}(Y) = \vec{0}$ ).

The minimal resolution  $X_k \rightarrow \mathbb{C}^2/\mathbb{Z}_k$  induces a natural  $\mathbb{T}^2$ -equivariant morphism  $\text{Hilb}^n(X_k) \rightarrow \text{Sym}^n(\mathbb{C}^2/\mathbb{Z}_k)$ . It is shown in [91, §8.1] that the generating function for equivariant volumes of the Hilbert scheme can be summed explicitly to give (cf. (2.14))

$$\sum_{n=0}^{\infty} \mathbf{q}^n \int_{\text{Hilb}^n(X_k)} 1 = \exp\left(\frac{\mathbf{q}}{k\epsilon_1\epsilon_2}\right).$$

The coefficient of  $\mathbf{q}^n$  is given by

$$\int_{\text{Hilb}^n(X_k)} 1 = \frac{1}{n! k^n (\epsilon_1 \epsilon_2)^n},$$

which is the expected  $\mathbb{T}^2$ -equivariant volume of the  $n$ -th symmetric product  $\text{Sym}^n(\mathbb{C}^2/\mathbb{Z}_k)$ . Using geometric arguments on quiver varieties, Fujii and Minabe show in [47] that  $\text{Hilb}^{|\vec{v}|}(\mathbb{C}^2)^{\mathbb{Z}_k, \vec{v}} \rightarrow \text{Sym}^n(\mathbb{C}^2/\mathbb{Z}_k)$  is a  $\mathbb{T}^2$ -equivariant resolution of singularities for  $n = |\vec{v}|$ , and thus  $\int_{\text{Hilb}^{|\vec{v}|}(\mathbb{C}^2)^{\mathbb{Z}_k, \vec{v}}} 1$  also gives the equivariant volume of  $\text{Sym}^n(\mathbb{C}^2/\mathbb{Z}_k)$ . Hence the series over  $k$ -hooks can be summed explicitly to give [47, Prop. A.5]

$$\sum_{\nu_i(Y)=v_i} \prod_{s \in \vec{q}(Y)} \frac{1}{((L(s)+1)\epsilon_1 - A(s)\epsilon_2)(-L(s)\epsilon_1 + (A(s)+1)\epsilon_2)} = \frac{1}{n! k^n (\epsilon_1 \epsilon_2)^n}$$

where  $n = |\vec{q}(Y)|$ . It follows that

$$\mathcal{Z}_{[\mathbb{C}^2/\mathbb{Z}_k]}(\epsilon_1, \epsilon_2; \tilde{\mathbf{q}}, \vec{\mathbf{q}})^{\text{quot}} = \sum_{n=0}^{\infty} \tilde{\mathbf{q}}^n k^n \vec{\mathbf{q}}^n \frac{1}{n! k^n (\epsilon_1 \epsilon_2)^n} = \exp\left(\frac{\mathbf{q}}{k\epsilon_1\epsilon_2}\right)$$

where

$$\mathbf{q} := \tilde{\mathbf{q}}^k \prod_{i=0}^{k-1} \mathbf{q}_i.$$

For the  $k$ -core contribution, we identify the set of  $k$ -cores  $(\mathbb{Z}^I)_0$  with the  $A_{k-1}$  root lattice  $\mathcal{Q}$ , and proceed similarly to the proof of [47, Lem. 4.9] to compute

$$\begin{aligned} \mathcal{Z}_{[\mathbb{C}^2/\mathbb{Z}_k]}(\tilde{\mathbf{q}}, \vec{\mathbf{q}})^{\text{core}} &= \sum_{c \in \mathcal{Q}} \mathbf{q}^{\frac{1}{2} \sum_{j=0}^{k-1} c_{j+\frac{1}{2}}^2} \prod_{i=1}^{k-1} (\tilde{\mathbf{q}} \mathbf{q}_i)^{\sum_{j=i}^{k-1} c_{j+\frac{1}{2}}} \\ &= \sum_{c \in \mathcal{Q}} \mathbf{q}^{\frac{1}{2} \sum_{j=0}^{k-1} (v_j(c) - v_{j+1}(c))^2} \prod_{i=1}^{k-1} (\tilde{\mathbf{q}} \mathbf{q}_i)^{\sum_{j=i}^{k-1} (v_j(c) - v_{j+1}(c))} \\ &= \sum_{c \in \mathcal{Q}} \mathbf{q}^{\frac{1}{2} \vec{v}(c) \cdot \widehat{C} \vec{v}(c)} \prod_{i=1}^{k-1} (\tilde{\mathbf{q}} \mathbf{q}_i)^{v_i(c) - v_0(c)}. \end{aligned}$$

Here we use the fact that, for any element  $c := (c_{\frac{1}{2}}, \dots, c_{k-\frac{1}{2}})$  of the root lattice  $\mathcal{Q}$ , there is a unique vector  $\vec{v} = (v_0, v_1, \dots, v_{k-1}) \in \mathbb{Z}_{\geq 0}^k$  such that  $c_{\frac{2i+1}{2}} = v_i \bmod k - v_{i+1} \bmod k$  for  $i = 0, 1, \dots, k-1$  and

$\frac{1}{2} \vec{v} \cdot \widehat{C} \vec{v} + v_0 \in \mathbb{Z}_{>0}$ . To make contact with the partition function on the resolution from §2.5, we introduce the first Chern numbers  $\vec{u}(c) = -\widehat{C} \vec{v}(c)$  as before, and define  $\vec{w}(c) = (w_1(c), \dots, w_{k-1}(c))$  by  $w_i(c) = v_i(c) - v_0(c)$  for  $i = 1, \dots, k-1$ . Then  $\vec{v}(c) \cdot \widehat{C} \vec{v}(c) = \vec{w}(c) \cdot C \vec{w}(c)$  and  $u_i(c) = -\sum_{j=1}^{k-1} C_{ij} w_j(c)$  for  $i = 1, \dots, k-1$ . It follows that

$$\mathcal{Z}_{[\mathbb{C}^2/\mathbb{Z}_k]}(\tilde{\mathbf{q}}, \vec{\mathbf{q}})^{\text{core}} = \sum_{c \in \mathbb{Q}} \mathbf{q}^{\frac{1}{2} \sum_{i,j=1}^{k-1} u_i(c) (C^{-1})^{ij} u_j(c)} \prod_{i=1}^{k-1} \xi_i^{u_i(c)},$$

where

$$\xi_i := \prod_{j=1}^{k-1} (\tilde{\mathbf{q}} \mathbf{q}_j)^{(C^{-1})^{ij}}$$

for  $i = 1, \dots, k-1$ .

Putting everything together, we have thus shown that the gauge theory partition functions on the quotient stack and on the minimal resolution in this case coincide:

$$\mathcal{Z}_{[\mathbb{C}^2/\mathbb{Z}_k]}^{\text{inst}}(\epsilon_1, \epsilon_2; \tilde{\mathbf{q}}, \vec{\mathbf{q}}) = \mathcal{Z}_{X_k}^{\text{inst}}(\epsilon_1, \epsilon_2; \mathbf{q}, \vec{\xi})_{\vec{w}_0}.$$

The Nekrasov partition functions in the generic case of  $U(N)$  gauge theory on  $[\mathbb{C}^2/\mathbb{Z}_k]$  can be similarly defined and regarded as  $\widetilde{\mathbb{T}}$ -equivariant integrals over the moduli spaces  $\mathfrak{M}_{\xi_0}(\vec{v}, \vec{w})$ , see e.g. [113, 11, 5, 60, 24]; the  $\widetilde{\mathbb{T}}$ -fixed points in this case are vectors of  $\vec{w}$ -coloured Young diagrams  $\vec{Y}$  with  $k$  colours. However, in this case the torus fixed point sets  $\mathfrak{M}_{\xi_0}(\vec{v}, \vec{w})^{\widetilde{\mathbb{T}}}$  and  $\mathfrak{M}_{\xi_\infty}(\vec{v}, \vec{w})^{\widetilde{\mathbb{T}}}$  are not in bijective correspondence, and the explicit matching with the partition function on the minimal resolution  $X_k$  is not presently understood in generality.

## 2.7 $\mathcal{N} = 1$ gauge theory in five dimensions

Let us now sketch how to derive our  $\mathcal{N} = 2$  gauge theory partition functions directly from the six-dimensional  $\mathcal{N} = 1$  supersymmetric gauge theory defining the  $\Omega$ -background, as discussed in §2.2. We consider the limit in which the torus  $T^2$  collapses to a circle  $S^1$  of radius  $\varrho$ , which yields five-dimensional  $\mathcal{N} = 1$  gauge theory on the flat affine bundle  $M_\varrho \rightarrow S^1$  where  $M_\varrho := (\mathbb{C}^2 \times \mathbb{R})/\mathbb{Z}$  with the  $\mathbb{Z}$ -action given by

$$n \triangleright (z_1, z_2, x) = (q^n z_1, t^n z_2, x + 2\pi n \varrho).$$

Here  $n \in \mathbb{Z}$  is the instanton charge of the four-dimensional description,  $(z_1, z_2) \in \mathbb{C}^2$ ,  $x \in \mathbb{R}$ , and  $(q, t)$  is an element of the maximal torus of the complex rotation group  $GL(2, \mathbb{C})$  which parameterizes the holonomy of a connection on the flat  $\mathbb{T}^2$ -bundle discussed in §2.2. Then we take the collapsing limit  $\varrho \rightarrow 0$  to get four-dimensional  $\mathcal{N} = 2$  gauge theory on the  $\Omega$ -background. We shall study the Nekrasov partition function for  $\mathcal{N} = 2$  gauge theory on  $[\mathbb{C}^2/\mathbb{Z}_k]$  by studying the Uglov limits of the corresponding gauge theories on  $M_\varrho$  (see §B.3). Partition functions of gauge theories on  $M_\varrho$  also compute the K-theory versions of Nekrasov partition functions on  $\mathbb{C}^2$ .

Let  $\varrho \in \mathbb{C}$  be a parameter. For a vector bundle  $V$  over  $\text{Hilb}^n(\mathbb{C}^2)$ , let  $\widehat{A}_\varrho(V)$  be the deformation of the  $\widehat{A}$ -genus of  $V$  defined in [51, §4.6]. The instanton partition function of pure  $\mathcal{N} = 1$  five-dimensional gauge theory on  $M_\varrho$  is defined by

$$\mathcal{Z}_{M_\varrho}^{\text{inst}}(q, t; \mathbf{q}) := \sum_{n=0}^{\infty} \mathbf{q}^n \int_{\text{Hilb}^n(\mathbb{C}^2)} \widehat{A}_\varrho(T\text{Hilb}^n(\mathbb{C}^2)).$$

By the localization theorem one has

$$\int_{\text{Hilb}^n(\mathbb{C}^2)} \hat{A}_\varrho(T\text{Hilb}^n(\mathbb{C}^2)) = \sum_{|\lambda|=n} \prod_{s \in Y_\lambda} \frac{\varrho^2}{(1 - q^{-A(s)-1} t^{-L(s)}) (1 - q^{-A(s)} t^{-L(s)-1})}, \quad (2.15)$$

where we introduced the parameters

$$q = t_2^\varrho = e^{\varrho \epsilon_2} \quad \text{and} \quad t = t_1^{-\varrho} = e^{-\varrho \epsilon_1} = q^\beta$$

with  $\beta = -\epsilon_1/\epsilon_2$ . This partition function is a  $q$ -deformation of the partition function (2.2) for four-dimensional pure  $\mathcal{N} = 2$  gauge theory on  $\mathbb{C}^2$ , to which it reduces in the collapsing limit  $\varrho \rightarrow 0$ ,

$$\lim_{\varrho \rightarrow 0} \mathcal{Z}_{M_\varrho}^{\text{inst}}(q = t_2^\varrho, t = q^\beta; \mathbf{q}) = \mathcal{Z}_{\mathbb{C}^2}^{\text{inst}}(\epsilon_1, \epsilon_2; \mathbf{q}).$$

In [82, §4] the same expression is derived as the K-theory version of the Nekrasov partition function on  $\mathbb{C}^2$ , wherein we replace integration in equivariant cohomology by integration in equivariant K-theory. The series over partitions can be explicitly summed (see e.g. [82, §4]) to get

$$\mathcal{Z}_{M_\varrho}^{\text{inst}}(q, t; \mathbf{q}) = \prod_{i,j=0}^{\infty} \frac{1}{1 - t^i q^{-j} \mathbf{q} \varrho^2} = \exp \left( \sum_{r=1}^{\infty} \frac{\mathbf{q}^r \varrho^{2r}}{r (1 - q^{-r}) (1 - t^r)} \right).$$

This formula can be derived from the Cauchy–Stanley identity for Macdonald symmetric functions, see §B.2; it reduces to (2.3) in the limit  $\varrho \rightarrow 0$ . In the following we regard  $\mathcal{Z}_{M_\varrho}^{\text{inst}}(q, t; \mathbf{q})$  as a function of arbitrary  $q, t \in \mathbb{C}$ .

We will now consider this partition function in the *Uglov limit* by instead setting

$$q = \omega e^{\varrho \epsilon_2} = \omega p \quad \text{and} \quad t = \omega e^{-\varrho \epsilon_1} = \omega p^\beta \quad \text{with} \quad p = t_2^\varrho,$$

and defining the orbifold partition function as

$$\mathcal{Z}_{[\mathbb{C}^2/\mathbb{Z}_k]}^{\text{inst}}(\epsilon_1, \epsilon_2; \mathbf{q}) := \lim_{\varrho \rightarrow 0} \mathcal{Z}_{M_\varrho}^{\text{inst}}(q = \omega t_2^\varrho, t = \omega t_1^{-\varrho}; \mathbf{q}).$$

Multiplication with the primitive  $k$ -th root of unity  $\omega := e^{2\pi i/k}$  implements the orbifold projection onto  $\mathbb{Z}_k$ -invariant sectors [65]. For each partition  $\lambda$  occurring in the bilinear form product (2.15), one easily sees that the product gives 0 in the limit  $\varrho \rightarrow 0$  unless  $\omega^{h(s)} = 1$  for all nodes  $s \in Y_\lambda$ . Thus the series truncates to a sum over  $k$ -coloured Young diagrams and one has

$$\mathcal{Z}_{[\mathbb{C}^2/\mathbb{Z}_k]}^{\text{inst}}(\epsilon_1, \epsilon_2; \mathbf{q}) = \sum_{\lambda} \mathbf{q}^{|\lambda|} \prod_{\substack{s \in Y_\lambda \\ h(s) \equiv 0 \pmod k}} \prod_{s \in Y_\lambda} \frac{1}{((L(s)+1)\epsilon_1 - A(s)\epsilon_2) (-L(s)\epsilon_1 + (A(s)+1)\epsilon_2)},$$

which agrees with the partition function of §2.6 at  $\tilde{\mathbf{q}} = \mathbf{q}$  and  $\mathbf{q}_i = 1$  for  $i = 0, 1, \dots, k-1$ .

## 2.8 Quantization of coarse moduli spaces

In the remainder of this section we shall initiate some of the geometric constructions underlying a putative reformulation of the partition functions of  $\mathcal{N} = 2$  gauge theories on ALE spaces in terms of contributions from noncommutative instantons. The Pontrjagin dual group  $\widehat{\mathbb{T}^2} = \text{Hom}_{\mathbb{C}}(\mathbb{T}^2, \mathbb{C}^\times) \cong \mathbb{Z} \oplus \mathbb{Z}$  is the group of characters  $\{\chi_p\}_{p \in L^*}$  parameterized by elements of the dual lattice  $p = p_1 e_1^* + p_2 e_2^* \in L^*$  as

$$\chi_p(t) = t^p := t_1^{p_1} t_2^{p_2}. \quad (2.16)$$

The unital algebra  $\mathcal{H} = \mathcal{A}(\mathbb{T}^2)$  of coordinate functions on the torus  $\mathbb{T}^2$  is the Laurent polynomial algebra

$$\mathcal{H} := \mathbb{C}(t_1, t_2)$$

generated by elements  $t_1, t_2$ . It is equipped with the Hopf algebra structure

$$\Delta(t^p) = t^p \otimes t^p, \quad \varepsilon(t^p) = 1 \quad \text{and} \quad S(t^p) = t^{-p}$$

for  $p \in L^*$ , with the coproduct and the counit respectively extended as algebra morphisms  $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$  and  $\varepsilon : \mathcal{H} \rightarrow \mathbb{C}$ , and the antipode as an anti-algebra morphism  $S : \mathcal{H} \rightarrow \mathcal{H}$ .

The canonical right action of  $\mathbb{T}^2$  on itself by group multiplication dualizes to give a left  $\mathcal{H}$ -coaction

$$\Delta_L : \mathcal{A}(\mathbb{T}^2) \longrightarrow \mathcal{H} \otimes \mathcal{A}(\mathbb{T}^2), \quad \Delta_L(u_i) = t_i \otimes u_i, \quad \Delta_L(u_i^{-1}) = t_i^{-1} \otimes u_i^{-1}, \quad (2.17)$$

where we write  $u_i, u_i^{-1}$ ,  $i = 1, 2$ , for the generators of  $\mathcal{A}(\mathbb{T}^2)$  viewed as a left comodule algebra over itself, when distinguishing the coordinate algebra  $\mathcal{A}(\mathbb{T}^2)$  from the Hopf algebra  $\mathcal{H}$ . This coaction is equivalent to a grading of the algebra  $\mathcal{A}(\mathbb{T}^2)$  by the dual lattice  $L^*$ , for which the homogeneous elements are the characters (2.16).

The functorial quantization constructed in [34, §1.2] twists the algebra multiplication in  $\mathcal{A} = \mathcal{A}(\mathbb{T}^2)$  into a new product. Let  $\mathcal{A}(\mathbb{T}_\theta^2)$  be the Laurent polynomial algebra generated by  $u_1$  and  $u_2$  with this product, where  $\theta \in \mathbb{C}$  is the deformation parameter. It has relations

$$u_1^{p_1} u_2^{p_2} = q^{2p_1 p_2} u_2^{p_2} u_1^{p_1} \quad \text{for } p_1, p_2 \in \mathbb{Z}, \quad (2.18)$$

where  $q = \exp(\frac{i}{2}\theta)$ . This quantizes the torus  $\mathbb{T}^2$  into the noncommutative complex torus  $\mathbb{T}_\theta^2$  dual to the algebra  $\mathcal{A}(\mathbb{T}_\theta^2)$ . We will sometimes write  $\mathcal{A}(\mathbb{T}_\theta^2) = \mathbb{C}_\theta(u_1, u_2)$  for this coordinate algebra.

Noncommutative affine toric varieties correspond to finitely-generated  $\mathcal{H}_\theta$ -comodule subalgebras of the algebra  $\mathcal{A}(\mathbb{T}_\theta^2)$  of the noncommutative torus [33, §3], where  $\mathcal{H}_\theta$  is the Hopf algebra  $\mathcal{H}$  with twisted coquasitriangular structure  $\mathcal{R}_\theta : \mathcal{H}_\theta \otimes \mathcal{H}_\theta \rightarrow \mathbb{C}$  defined on generators by  $\mathcal{R}_\theta(t_i \otimes t_i) = 1$  for  $i = 1, 2$  and  $\mathcal{R}_\theta(t_1 \otimes t_2) = q^{-2} = \mathcal{R}_\theta(t_2 \otimes t_1)^{-1}$ . To each cone  $\sigma \subset L_\mathbb{R}$ , we define the coordinate algebra  $\mathbb{C}_\theta[\sigma]$  dual to a noncommutative affine variety  $U_\theta[\sigma]$  to be the subalgebra of  $\mathcal{A}(\mathbb{T}_\theta^2)$  spanned by the Laurent monomials  $u^{m_a}$ ,  $a = 1, \dots, r \geq 2$ , subject to relations derived from (2.18), where  $m_a$  are the generators of the semigroup  $\check{\sigma} \cap L^*$  and

$$\check{\sigma} = \{m \in L_\mathbb{R}^* \mid \langle m, v \rangle \geq 0 \quad \forall v \in \sigma\}$$

is the dual cone to  $\sigma$ . We will sometimes denote this noncommutative algebra by  $\mathbb{C}_\theta[\sigma] = \mathbb{C}_\theta[u^{m_1}, \dots, u^{m_r}]$ .

Let us begin by spelling out the noncommutative toric geometry of the compactified orbit space; we write  $\mathcal{A}' = \mathcal{A}((\mathbb{P}^2/\mathbb{Z}_k)_\theta)$  for the coordinate algebra. For the two-cones  $\sigma'_{0,k} := \mathbb{R}_{\geq 0}v_0 + \mathbb{R}_{\geq 0}v_k$ ,  $\sigma'_{0,\infty} := \mathbb{R}_{\geq 0}v_0 + \mathbb{R}_{\geq 0}(-v_0 - v_k)$ , and  $\sigma'_{k,\infty} := \mathbb{R}_{\geq 0}v_k + \mathbb{R}_{\geq 0}(-v_0 - v_k)$ , the relations among the generators of the subalgebras  $\mathbb{C}_\theta[\sigma'_{i,j}] \subset \mathcal{A}(\mathbb{T}_\theta^2)$  are as follows:

- (O1) The semigroup  $\check{\sigma}'_{0,k} \cap L^*$  is generated over  $\mathbb{Z}_{\geq 0}$  by the lattice vectors  $m'_1 = (k-1)e_1^* + ke_2^*$ ,  $m'_2 = e_1^*$ , and  $m'_3 = e_1^* + e_2^*$ , with the relation  $m'_1 + m'_2 = km'_3$ . The coordinate algebra  $\mathbb{C}_\theta[\sigma'_{0,k}]$  of the noncommutative affine variety  $U_\theta[\sigma'_{0,k}]$  is thus the polynomial algebra  $\mathbb{C}[x, y, z]$  generated over  $\mathbb{C}$  by  $x = u^{m'_1} = u_1^{k-1}u_2^k$ ,  $y = u^{m'_2} = u_1$ , and  $z = u^{m'_3} = u_1u_2$  subject to the commutation relations

$$xy = q^{-2k}yx, \quad xz = q^{-2}zx \quad \text{and} \quad yz = q^2zy \quad (2.19)$$



together with

$$x y - q^{k(k-3)} z^k = 0 . \quad (2.20)$$

These relations demonstrate that  $U_\theta[\sigma'_{0,k}]$  is a toric noncommutative deformation of the  $A_{k-1}$  orbifold singularity  $\mathbb{C}^2/\mathbb{Z}_k$ ; this generalizes the  $k = 2$  construction of [33, §3.4].

- (O2) The semigroup  $\sigma'_{0,\infty} \cap L^*$  is generated by  $m'_1 = (k-2)e_1^* + k e_2^*$ ,  $m'_2 = e_1^*$ , and  $m'_3 = (k-1)e_1^* + k e_2^*$  with the relation  $m'_1 + m'_2 = m'_3$ . The coordinate algebra  $\mathbb{C}_\theta[\sigma'_{0,\infty}]$  of  $U_\theta[\sigma'_{0,\infty}]$  is generated by  $x = u_1^{k-2} u_2^{-2}$ ,  $y = u_1$ , and  $z = u_1^{k-1} u_2^k$  with the commutation relations

$$x y = q^{-2k} y x , \quad x z = q^{-2k} z x \quad \text{and} \quad y z = q^{2k} z y \quad (2.21)$$

together with

$$x y - q^{-2k} z = 0 .$$

- (O3) The semigroup  $\sigma'_{k,\infty} \cap L^*$  is generated by exactly the same lattice vectors as in item (O2), and hence the coordinate algebra  $\mathbb{C}_\theta[\sigma'_{k,\infty}]$  of  $U_\theta[\sigma'_{k,\infty}]$  also has the same generators and relations.

Away from the orbifold points, we can glue these subalgebras of  $\mathcal{A}(\mathbb{T}_\theta^2)$  together via algebra automorphisms in the braided monoidal category  ${}^{\mathcal{H}_\theta}\mathcal{M}$  of left  $\mathcal{H}_\theta$ -comodules to form the global noncommutative toric orbit space  $(\mathbb{P}^2/\mathbb{Z}_k)_\theta$  [33, §3]. We denote the orbit one-cones by  $\tau'_0 = \mathbb{R}_{\geq 0} v_0$ ,  $\tau'_k = \mathbb{R}_{\geq 0} v_k$ , and  $\tau'_\infty = \mathbb{R}_{\geq 0}(-v_0 - v_k)$ . Their quantization is described as follows:

- For the face  $\tau'_0 = \sigma'_{0,k} \cap \sigma'_{0,\infty}$ , the semigroup  $\tau'_0 \cap L^*$  is generated by  $m'_1 = e_2^*$ ,  $m'_2 = e_1^*$ , and  $m'_3 = -e_1^* = -m'_2$ . The generators over  $\mathbb{C}$  of the subalgebra  $\mathbb{C}_\theta[\tau'_0] = \mathbb{C}_\theta[s, t, t^{-1}]$  are  $s = u_1^{m'_1} = u_2$  and  $t = u_1^{m'_2} = u_1$  with the relations

$$s t = q^{-2} t s , \quad s t^{-1} = q^2 t^{-1} s \quad \text{and} \quad t t^{-1} = 1 = t^{-1} t . \quad (2.22)$$

These relations show that the noncommutative affine variety  $U_\theta[\tau'_0] \cong \mathbb{P}_\theta^1$  is that of a toric noncommutative projective line [33]. Recalling the generators of  $\mathbb{C}_\theta[\sigma'_{0,k}]$  and  $\mathbb{C}_\theta[\sigma'_{0,\infty}]$ , the inclusions  $\tau'_0 \hookrightarrow \sigma'_{0,k}$  and  $\tau'_0 \hookrightarrow \sigma'_{0,\infty}$  induce canonical morphisms in  ${}^{\mathcal{H}_\theta}\mathcal{M}$  of noncommutative algebras  $\mathbb{C}_\theta[\sigma'_{0,k}] \rightarrow \mathbb{C}_\theta[\tau'_0]$  and  $\mathbb{C}_\theta[\sigma'_{0,\infty}] \rightarrow \mathbb{C}_\theta[\tau'_0]$  which are both natural inclusions of subalgebras.

- For the face  $\tau'_k = \sigma'_{0,k} \cap \sigma'_{k,\infty}$ , the semigroup  $\tau'_k \cap L^*$  is generated by  $m'_1 = e_2^*$ ,  $m'_2 = e_1^* - k e_2^*$ , and  $m'_3 = k e_2^* - e_1^* = -m'_2$ . The generators of  $\mathbb{C}_\theta[\tau'_k] = \mathbb{C}_\theta[s, t, t^{-1}]$  are  $s = u_2$  and  $t = u_1 u_2^{-k}$ , again with the commutation relations (2.22). There are natural subalgebra inclusion morphisms  $\mathbb{C}_\theta[\sigma'_{0,k}] \rightarrow \mathbb{C}_\theta[\tau'_k]$  and  $\mathbb{C}_\theta[\sigma'_{k,\infty}] \rightarrow \mathbb{C}_\theta[\tau'_k]$ .
- The face  $\tau'_\infty = \sigma'_{0,\infty} \cap \sigma'_{k,\infty}$  is the orbit “line at infinity”. The semigroup  $\tau'_\infty \cap L^*$  is generated by  $m'_1 = e_2^*$ ,  $m'_2 = (k-2)e_1^* + k e_2^*$ ,  $m'_3 = -(k-2)e_1^* - k e_2^* = -m'_2$ , and  $m'_4 = -e_2^* = -m'_1$ . The generators of the subalgebra  $\mathbb{C}_\theta[\tau'_\infty] = \mathbb{C}_\theta(s, t)$  are  $s = u_2$  and  $t = u_1^{k-2} u_2^k$  with the relations

$$s t = q^{-2(k-2)} t s \quad \text{and} \quad s t^{-1} = q^{2(k-2)} t^{-1} s$$

together with

$$s s^{-1} = 1 = s^{-1} s \quad \text{and} \quad t t^{-1} = 1 = t^{-1} t .$$

Again there are natural subalgebra inclusion morphisms  $\mathbb{C}_\theta[\sigma'_{0,\infty}] \rightarrow \mathbb{C}_\theta[\tau'_\infty]$  and  $\mathbb{C}_\theta[\sigma'_{k,\infty}] \rightarrow \mathbb{C}_\theta[\tau'_\infty]$ . Note that for  $k = 2$  the algebra  $\mathbb{C}_\theta(s, t)$  describes a *commutative* orbit line.

Let us now turn to the noncommutative toric resolution; we write  $\bar{\mathcal{A}} = \mathcal{A}((\overline{X_k})_\theta)$  for the dual coordinate algebra. The generators of the subalgebras  $\mathbb{C}_\theta[\sigma'_{i,\infty}]$ ,  $i = 0, k$ , associated to the two-cones of the orbit compactification divisor  $[\ell_\infty/\mathbb{Z}_k]$  are described by items (O2) and (O3) above, while those of the subalgebras  $\mathbb{C}_\theta[\sigma_i] \subset \mathcal{A}(\mathbb{T}_\theta^2)$  associated to the two-cones  $\sigma_i = \mathbb{R}_{\geq 0}v_{i-1} + \mathbb{R}_{\geq 0}v_i$ ,  $i = 1, \dots, k$ , of  $\overline{X_k}$  are described as follows. The generators of the semigroup  $\sigma_i \cap L^*$  over  $\mathbb{Z}_{\geq 0}$  are  $\bar{m}_1 = -(i-2)e_1^* - (i-1)e_2^*$  and  $\bar{m}_2 = (i-1)e_1^* + ie_2^*$  for each  $i = 1, \dots, k$ . In this case the algebra  $\mathbb{C}_\theta[\sigma_i] = \mathbb{C}_\theta[\bar{x}, \bar{y}]$  is generated over  $\mathbb{C}$  by  $\bar{x} = u^{\bar{m}_1} = u_1^{i-2}u_2^{i-1}$  and  $\bar{y} = u^{\bar{m}_2} = u_1^{-(i-1)}u_2^{-i}$  with the quadratic relations

$$\bar{x}\bar{y} = q^2 \bar{y}\bar{x}.$$

It follows that the noncommutative affine varieties  $U_\theta[\sigma_i] \cong \mathbb{C}_\theta^2$  for  $i = 1, \dots, k$  are each copies of the two-dimensional algebraic Moyal plane [33].

The “inner” one-cones of the toric resolution are denoted  $\tau_i := \mathbb{R}_{\geq 0}v_i = \sigma_i \cap \sigma_{i+1}$  for  $i = 1, \dots, k-1$ . The semigroup  $\tau_i \cap L^*$  is generated by  $\bar{m}_1 = e_2^*$ ,  $\bar{m}_2 = e_1^* - ie_2^*$ , and  $\bar{m}_3 = ie_2^* - e_1^* = -\bar{m}_2$ . The generators of the noncommutative coordinate algebra  $\mathbb{C}_\theta[\tau_i] = \mathbb{C}_\theta[s, t, t^{-1}]$  are  $s = u_2$  and  $t = u_1 u_2^{-i}$  with the relations (2.22) of the noncommutative projective line  $\mathbb{P}_\theta^1$ . Again there are natural subalgebra inclusion morphisms  $\mathbb{C}_\theta[\sigma_i] \rightarrow \mathbb{C}_\theta[\tau_i]$  and  $\mathbb{C}_\theta[\sigma_{i+1}] \rightarrow \mathbb{C}_\theta[\tau_i]$ , and in  $\mathbb{C}_\theta[\tau_i]$  a natural algebra automorphism  $\mathbb{C}_\theta[\sigma_i] \rightarrow \mathbb{C}_\theta[\sigma_{i+1}]$  is given on generators by  $(u_1, u_2) \mapsto (u_1 u_2^{-1}, u_2)$  for each  $i = 1, \dots, k-1$ ; this is similar to the automorphisms defining the toric noncommutative projective plane  $\mathbb{P}_\theta^2$  [33, §3.3]. The “outer” one-cones  $\tau'_0 = \sigma_1 \cap \sigma'_{0,\infty}$  and  $\tau'_k = \sigma_k \cap \sigma'_{k,\infty}$ , as well as the orbit “line at infinity”  $\tau'_\infty$ , are treated as above.

## 2.9 Noncommutative instantons on quotient stacks

To proceed now with a construction of moduli spaces of noncommutative instantons, we need a better “global” description of these toric deformations as in [33, 34] which is moreover a quantization of the overarching quotient groupoids. An extension of the Nekrasov–Schwarz noncommutative ADHM construction [87] to the Kleinian singularities  $\mathbb{C}^2/\mathbb{Z}_k$  was given by Lazariou [69]. In particular, he obtains a noncommutative geometry interpretation of their minimal resolutions, which is natural from the point of view of Yang–Mills matrix models and the resolution of orbifold singularities by D-branes, and provides another interpretation of Nakajima’s quiver varieties of type  $\widehat{A}_{k-1}$  [78] as moduli spaces of  $\mathbb{Z}_k$ -equivariant noncommutative instantons. We seek an analogous interpretation for our deformations.

For this, we follow the construction of deformations of Kleinian singularities (and of Hilbert schemes of points in  $\mathbb{C}^2$ ) in terms of multi-homogeneous coordinate algebras which was considered in [52, 53, 22, 64]; their toric geometry aspects are studied in [76, 13]. Classically, the quotient stack  $[\mathbb{C}^2/\mathbb{Z}_k]$ , or the corresponding dual coordinate algebra  $\mathbb{C}[z_1, z_2] \rtimes \mathbb{Z}_k$ , is a noncommutative crepant resolution of the  $\mathbb{C}^2/\mathbb{Z}_k$  orbifold singularity. By the McKay correspondence, there is an equivalence between the derived category of coherent sheaves on the minimal crepant resolution  $\text{Hilb}^{\mathbb{Z}_k}(\mathbb{C}^2) \cong X_k$  of the quotient singularity  $\mathbb{C}^2/\mathbb{Z}_k$  and the derived category of finitely generated modules over  $\mathbb{C}[z_1, z_2] \rtimes \mathbb{Z}_k$ . This equivalence generalizes to the noncommutative setting: One can construct an algebra which simultaneously gives a noncommutative deformation of  $\mathbb{C}^2/\mathbb{Z}_k$  and of its minimal resolution of singularities  $\text{Hilb}^{\mathbb{Z}_k}(\mathbb{C}^2) \rightarrow \mathbb{C}^2/\mathbb{Z}_k$ ; in contrast to the classical McKay correspondence, which involves only derived categories, this is given by an equivalence of abelian categories of modules.

We shall apply this equivalence to the noncommutative toric deformation of the quotient stack  $[\mathbb{P}^2/\mathbb{Z}_k]$ . For this, we recall the construction of the homogeneous coordinate algebra  $\mathcal{A} = \mathcal{A}(\mathbb{P}_\theta^2)$  of the noncommutative projective plane  $\mathbb{P}_\theta^2$  from [33, 34, 35]. It is the graded polynomial algebra in

three generators  $w_i$ ,  $i = 0, 1, 2$  of degree one with the quadratic relations

$$w_0 w_i = w_i w_0 \quad \text{for } i = 1, 2 \quad \text{and} \quad w_1 w_2 = q^2 w_2 w_1. \quad (2.23)$$

The algebra  $\mathcal{A}$  is an Artin–Schelter regular algebra of global homological dimension three which is naturally an  $\mathcal{H}_\theta$ -comodule algebra. Each monomial  $w_i$  generates a left denominator set in  $\mathcal{A}$ , and the degree zero subalgebra of the left Ore localization of  $\mathcal{A}$  with respect to  $w_i$  is naturally isomorphic to the noncommutative coordinate algebra of the  $i$ -th maximal cone in the fan of  $\mathbb{P}_\theta^2$ , for each  $i = 0, 1, 2$ . The category of coherent sheaves on the noncommutative projective plane  $\mathbb{P}_\theta^2$  is equivalent to the category of finitely-generated graded  $\mathcal{A}$ -modules.

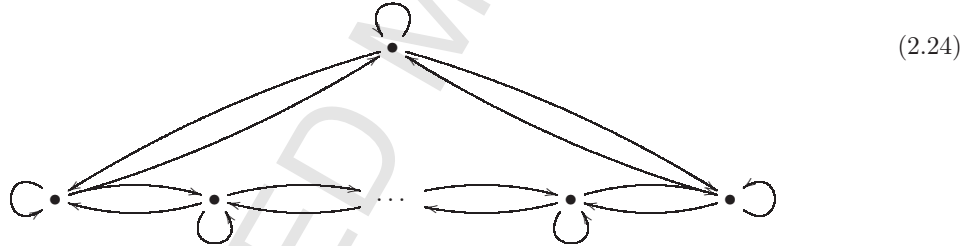
The cyclic group  $\mathbb{Z}_k$  acts by automorphisms of the graded algebra  $\mathcal{A}$  via (2.9). The subalgebra of invariants  $\mathcal{A}^{\mathbb{Z}_k} \subset \mathcal{A}$  is dual to the quantization of the orbit space  $\mathbb{P}^2/\mathbb{Z}_k$ ; it is generated by the central element  $w_0$  together with  $x := w_1^k$ ,  $y := w_2^k$  and  $z := w_1 w_2$  subject to the commutation relations

$$x y = q^{2k^2} y x, \quad x z = q^{2k} z x \quad \text{and} \quad y z = q^{-2k} z y$$

along with

$$x y - q^{k(k-1)} z^k = 0.$$

Just as in the commutative case, we can construct the McKay quiver  $\mathbf{Q}$  associated to this  $\mathbb{Z}_k$ -action on  $\mathcal{A}$  by assigning vertices to the character group  $\{0, 1, \dots, k-1\}$  of the cyclic group and arrows  $w_i^{(l)}$ ,  $i = 0, 1, 2$ ,  $l = 0, 1, \dots, k-1$ , whose target vertex shifts the source vertex  $l$  by the character of the  $\mathbb{Z}_k$ -eigenvector  $w_i$  minimally generating  $\mathcal{A}$ ; hence  $w_0^{(l)} : l \rightarrow l$  are vertex loops, while  $w_1^{(l)} : l \rightarrow l+1$ ,  $w_2^{(l)} : l \rightarrow l-1$  and the quiver  $\mathbf{Q}$  is given by



The relations  $\mathbf{R}$  of the quiver are induced by the relations (2.23) of the algebra  $\mathcal{A}$ , where composition of arrows makes sense, giving

$$w_0^{(l+1)} w_1^{(l)} = w_1^{(l)} w_0^{(l)}, \quad w_0^{(l-1)} w_2^{(l)} = w_2^{(l)} w_0^{(l)}, \quad w_1^{(l-1)} w_2^{(l)} = q^2 w_2^{(l+1)} w_1^{(l)}. \quad (2.25)$$

Then the category of linear representations of the McKay quiver with relations  $(\mathbf{Q}, \mathbf{R})$  is equivalent to the category of  $\mathbb{Z}_k$ -equivariant  $\mathcal{A}$ -modules [97].

Alternatively, the toric noncommutative deformation of the quotient stack  $[\mathbb{P}^2/\mathbb{Z}_k]$  is dual to the skew-group algebra  $\mathcal{A} \rtimes \mathbb{Z}_k$  over the algebra  $\mathcal{A}$ , which replaces the algebra  $\mathcal{A}^{\mathbb{Z}_k}$ . It is the free  $\mathcal{A}$ -module  $\mathcal{A} \otimes \mathbb{C}[\mathbb{Z}_k]$  over the group algebra of the cyclic group  $\mathbb{Z}_k$ , with the twisted noncommutative product  $(a \otimes \xi) \cdot (a' \otimes \xi') := (a(\xi \triangleright a')) \otimes (\xi \xi')$  for  $a, a' \in \mathcal{A}$  and  $\xi, \xi' \in \mathbb{Z}_k$ ; it has generators  $w_i^{(l)}$  with the relations (2.25), and hence the representation category of the quiver  $(\mathbf{Q}, \mathbf{R})$  is equivalent to the category of modules over  $\mathcal{A} \rtimes \mathbb{Z}_k$ . This toric noncommutative deformation is analogous to the deformations of Kleinian singularities considered in [57, 71] in terms of crossed product  $C^*$ -algebras

and the twisted group algebra of  $\mathbb{Z}_k$  associated with a projective regular representation, which may be characterised in terms of noncommutative gauge theories and D-branes.

In fact, there is a stronger statement which generalises the well-known situation in the commutative case [97]: If  $A = A(Q, R)$  denotes the path algebra of the quiver  $(Q, R)$ , i.e. the algebra of  $\mathbb{C}$ -linear combinations of paths of the quiver with product defined by path concatenation whenever paths compose and 0 otherwise, modulo the ideal  $\langle R \rangle$  generated by the relations (2.25), then

$$A \cong \mathcal{A} \rtimes \mathbb{Z}_k .$$

In this sense the algebra  $\mathcal{A} \rtimes \mathbb{Z}_k$  plays the role of a “noncommutative crepant resolution” of the algebra of invariants  $\mathcal{A}^{\mathbb{Z}_k}$ . Note that localising with respect to the central element  $w_0$  reduces the quiver  $Q$  to the standard (unframed) McKay quiver associated to the Kleinian singularity  $\mathbb{C}^2/\mathbb{Z}_k$ , i.e. the double of the cyclic quiver (2.12) of type  $\hat{A}_{k-1}$ , and the relations  $R$  to a  $q$ -deformation of the moment map (2.10) (after framing) defining a Nakajima quiver variety of type  $\hat{A}_{k-1}$  which is obtained by replacing the commutator with the *braided commutator*  $[b_1, b_2]_\theta := b_1 b_2 - q^2 b_2 b_1$ . In this way, a construction of instanton moduli spaces over the toric noncommutative deformation of  $[\mathbb{P}^2/\mathbb{Z}_k]$  via a braided version of the ADHM construction should go through exactly as in [34], and as in [34, 35] the corresponding equivariant noncommutative instanton partition functions for  $N = 2$  gauge theory should coincide with their classical counterparts. It would be interesting to study further the properties of these new moduli spaces which are (commutative) deformations of the usual quiver varieties, as well as their relations with the noncommutative quiver varieties obtained via deformation quantization of their standard symplectic structure.

It is much more difficult to construct an analogous quantization of the resolution  $[\overline{X}_k]$  in terms of homogeneous coordinate algebras. The stacky fan  $\bar{\Sigma}_k = (L, \bar{\Sigma}_k, \beta_k)$  yields a description of  $[\overline{X}_k]$  as a global quotient stack in the following way [20] (see also [26]). There is a short exact sequence

$$0 \longrightarrow L^* \xrightarrow{\bar{\beta}_k^*} (\mathbb{Z}\bar{b}_1 \oplus \cdots \oplus \mathbb{Z}\bar{b}_{k+2})^* \xrightarrow{\bar{\beta}_k^\vee} \text{Pic}(\overline{X}_k) \longrightarrow 0$$

from which one can represent the Gale dual  $\bar{\beta}_k^\vee : (\mathbb{Z}^{k+2})^* \rightarrow \text{coker}(\bar{\beta}_k^*) = \mathbb{Z}^k \cong \text{Pic}(\overline{X}_k)$  of the map  $\bar{\beta}_k$ . Applying the exact functor  $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^\times)$  yields the exact sequence

$$1 \longrightarrow G_k \longrightarrow (\mathbb{C}^\times)^{k+2} \longrightarrow \mathbb{T}^2 \longrightarrow 1 ,$$

where  $G_k = \text{Hom}_{\mathbb{Z}}(\text{coker}(\bar{\beta}_k^*), \mathbb{C}^\times)$ . Then the toric orbifold  $[\overline{X}_k]$  is isomorphic to the stack quotient

$$[\overline{X}_k] \cong [Z_k / G_k] ,$$

where  $Z_k \subset \mathbb{C}^{k+2}$  is the union over  $(z_1, \dots, z_{k+2}) \in \mathbb{C}^{k+2}$  with pairwise neighbouring entries  $(z_i, z_{i+1}) \in \mathbb{T}^2$  for  $i = 1, \dots, k+2 \pmod{k+2}$ , and the abelian affine algebraic group  $G_k \subset (\mathbb{C}^\times)^{k+2}$  consists of torus points  $(t_1, \dots, t_{k+2})$  satisfying the relations

$$t_2 t_3^2 \cdots t_{k+1}^k = t_{k+2}^k = t_1 t_3 \cdots t_{k+1}^{k-1}$$

and acting on  $Z_k$  via the standard scaling action of  $(\mathbb{C}^\times)^{k+2}$  on  $\mathbb{C}^{k+2}$ ; the coarse moduli space  $\overline{X}_k$  is isomorphic to the GIT quotient  $Z_k/G_k$ . It is unclear whether or not the corresponding noncommutative coordinate algebras will have nice smoothness properties, or even how to deal with their higher homological dimension. For  $k = 2$ , isospectral deformations of the minimal resolution of the  $\mathbb{C}^2/\mathbb{Z}_2$  orbifold singularity are studied in [114] by regarding it as an Eguchi–Hanson space, i.e. the total space  $\mathcal{O}_{\mathbb{P}^1}(-2)$  of the canonical line bundle over  $\mathbb{P}^1$ .

### 3 $\mathcal{N} = (2, 0)$ superconformal theories in six dimensions

We now move on to the celebrated conjectural relations between certain two-dimensional conformal field theories and four-dimensional  $\mathcal{N} = 2$  gauge theories; from a mathematical perspective these dualities generally assert the existence of geometric highest weight representations of vertex algebras on the cohomology of moduli spaces of torsion-free sheaves. At present there are explicit results available only for the examples discussed in §2, so we focus our discussion to an overview of some of the results that have been obtained in those instances. We begin with the physical background to these conjectures as it is worthy of a brief account in order to understand properly the set-ups and origins of some of the statements.

#### 3.1 $\mathcal{N} = 2$ field theories of class $\mathcal{S}$

Let  $\Gamma$  be a finite subgroup of  $SU(2)$ . By the McKay correspondence,  $\Gamma$  can be identified with the Dynkin diagram of ADE type of a simply-laced Lie algebra  $\mathfrak{g}$ ; let  $G$  be the corresponding connected Lie group. The group  $\Gamma$  acts naturally on  $\mathbb{C}^2$  (viewed as the fundamental representation of  $SU(2)$ ), and isometrically on the standard round three-sphere  $S^3 \subset \mathbb{C}^2$ . We consider Type IIB string theory on the ten-manifold  $M \times \mathbb{C}^2/\Gamma$  which is the product of a Riemannian six-manifold  $M$  and the ADE singularity  $\mathbb{C}^2/\Gamma$ . Its reduction in the limit where  $S^3/\Gamma$  is collapsed to a point defines an  $\mathcal{N} = (2, 0)$  supersymmetric theory on  $M$  which is conformally invariant [109]. The six-dimensional  $(2, 0)$  theory is not a gauge theory but rather a quantum theory of gerbes (instead of principal bundles) with connection, and it has no known Lagrangian description. For the A-series, which we shall primarily focus on in this section, it provides the worldvolume description of M5-branes in M-theory [100]. Its partition function is a section of a vector bundle of rank  $r > 1$  over the moduli space of field configurations, i.e. the theory has a vector space of possible partition functions (instead of a unique one) called the space of *conformal blocks*. This is completely analogous to the well-known situation in two-dimensional conformal field theory, where partition functions are sections of Friedan–Shenker bundles over moduli spaces of punctured Riemann surfaces.

Consider now the topologically twisted  $(2, 0)$  theory on a six-manifold which is a product

$$M = X \times C ,$$

where as before  $X$  is a complex toric surface, and  $C$  is a Riemann surface of genus  $g$  with  $n$  marked points carrying labels  $(\rho_i, \mu_i)$  for  $i = 1, \dots, n$ ; here  $\rho_i : \mathfrak{su}_2 \rightarrow \mathfrak{g}$  are Lie algebra homomorphisms and  $\mu_i \in \mathbb{C}$  are “mass” parameters. Using conformal invariance one can consider two equivalent reductions of this theory. On the one hand, the limit where  $C$  collapses to a point defines a four-dimensional quantum field theory on  $X$  which, in addition to the labels  $(\rho_i, \mu_i)$ , depends only on the complex structure of the punctured Riemann surface  $C$  and its area. When  $\mu_i = 0$  for  $i = 1, \dots, n$ , this is an  $\mathcal{N} = 2$  supersymmetric gauge theory with structure group  $\prod_{i=1}^n G^{\rho_i}$  where  $G^{\rho_i}$  is the centralizer of  $\rho_i$  in  $G$ ; these are the  $\mathcal{N} = 2$  gauge theories in four dimensions of class  $\mathcal{S}$  [48, 50]. On the other hand, the limit where  $X$  collapses to a point defines a two-dimensional theory on  $C$ , and therefore we see an equality between a four-dimensional and a two-dimensional quantum field theory.

While this relation seems mysterious from a purely two-dimensional or four-dimensional perspective, it is not surprising from the six-dimensional point of view. The four-dimensional field theory of class  $\mathcal{S}$  is completely determined by the Riemann surface  $C$  and its punctures, and when it has  $\mathcal{N} = 2$  superconformal invariance the two-dimensional theory on  $C$  is a conformal field theory; the chiral fields on  $C$  arise as the zero modes of the  $(2, 0)$  tensor multiplet in six dimensions. In this instance the Seiberg–Witten curve  $\Sigma$ , which captures the low-energy dynamics of the  $\mathcal{N} = 2$  gauge theory [94], is a branched cover of  $C$ ; for example, when  $X = \mathbb{C}^2$  and  $\mathfrak{g} = \mathfrak{sl}_2$ ,  $\Sigma \subset T^*C$  is a double

cover of  $C$  of genus  $h = 3g - 3 + n$ . The geometry of  $\Sigma$  is completely encoded by the Seiberg–Witten prepotential  $\mathcal{F}_X(\vec{a}; \mathbf{q})$  which determines its periods

$$\tau_{W'} = -\frac{1}{2\pi i} \frac{\partial^2 \mathcal{F}_X}{\partial a_l \partial a_{l'}}(\vec{a}; \mathbf{q}) ,$$

and it describes the low-energy effective action of the  $\mathcal{N} = 2$  gauge theory. Different canonical homology bases for  $H_1(\Sigma, \mathbb{Z})/H_1(C, \mathbb{Z})$  are related by  $Sp(2h, \mathbb{Z})$  modular transformations which describe S-dualities in the low-energy physics. The four-dimensional perspective lends a multitude of new tricks and insights into two-dimensional conformal field theories, and conversely the two-dimensional perspective can naturally account for many gauge theoretic phenomena such as S-duality which is manifested as modular invariance of conformal field theory correlation functions.

More generally, Nekrasov’s partition function can be defined and studied for *any* theory of class  $\mathcal{S}$ , and it can be regarded as defining a two-dimensional holomorphic quantum field theory on the Riemann surface  $C$ . On general grounds, any  $\mathcal{N} = 2$  gauge theory in four dimensions can be naturally associated to an algebraic integrable system, called the Donagi–Witten integrable system [72, 39], such that the Seiberg–Witten curve  $\Sigma$  coincides with the spectral curve of an associated Hitchin system on  $C$ . The  $\Omega$ -deformed theories of class  $\mathcal{S}$  are quantizations of these Hitchin systems which are obtained in the limit  $\epsilon_1, \epsilon_2 \rightarrow 0$ . For  $X = \mathbb{C}^2$  this perspective was used in [88] to conjecturally capture the spectrum of a quantum integrable system in the limit  $\epsilon_2 \rightarrow 0$  of the  $\Omega$ -deformation; we return to this description in §4.

This construction is at the heart of recent 2d/4d dualities, and in particular the AGT and BPS/CFT correspondences. In this setting the (localized)  $\tilde{\mathbb{T}}$ -equivariant cohomology  $\mathcal{H}_N(X)$  of the instanton moduli space  $\mathfrak{M}_N(X) = \bigsqcup_{n \geq 0} \mathfrak{M}_{N,n}(X)$  conjecturally carries a geometric highest weight representation of the vertex algebra  $\mathcal{A}_N(X)$  underlying the conformal field theory on  $C$ . The Nekrasov partition functions for pure  $\mathcal{N} = 2$  gauge theory on  $X$  are conjecturally given as geometric inner products involving a Whittaker vector (or irregular conformal block) for this representation, while the Nekrasov functions for  $\mathcal{N} = 2$  quiver gauge theories on  $X$  should coincide with conformal blocks on  $C$ . In the remainder of this section we summarize the current state of affairs for some of these results in the classes of examples considered in §2.

### 3.2 AGT duality on $\mathbb{C}^2$

Let us look first at the case  $X = \mathbb{C}^2$ , where the holomorphic field theory on  $C$  is believed to be the theory of  $\mathcal{W}(\mathfrak{gl}_N)$  conformal blocks (see §C). Consider the infinite-dimensional  $\mathbb{Z}_{\geq 0}$ -graded vector space

$$\mathcal{H}_N(\mathbb{C}^2) = \bigoplus_{n=0}^{\infty} \mathcal{H}_{N,n}(\mathbb{C}^2) := \bigoplus_{n=0}^{\infty} H_{\tilde{\mathbb{T}}}^{\bullet}(\mathfrak{M}_{N,n}(\mathbb{C}^2))_{\text{loc}} ,$$

where the subscript designates the *localized* equivariant cohomology which is the corresponding  $\mathbb{F}$ -vector space, with  $\mathbb{F}$  the quotient field of the coefficient ring  $H_{\tilde{\mathbb{T}}}^{\bullet}(\text{pt})$ . Let  $\iota_{\vec{Y}}$  denote the inclusion in  $\mathfrak{M}_{N,n}(\mathbb{C}^2)$  of the fixed point parameterized by the  $N$ -vector of Young diagrams  $\vec{Y}$ , and define distinguished classes  $[\vec{Y}] := \iota_{\vec{Y}*}(1)$  in  $H_{\tilde{\mathbb{T}}}^{\bullet}(\mathfrak{M}_{N,n}(\mathbb{C}^2))$ . Using the projection formula

$$[\vec{Y}] \cup [\vec{Y}'] = \delta_{\vec{Y}, \vec{Y}'} \text{Eu}(T_{\vec{Y}} \mathfrak{M}_{N,n}(\mathbb{C}^2))^{-1} [\vec{Y}]$$

for the cup product in equivariant cohomology, one defines a non-degenerate symmetric bilinear form  $\langle -, - \rangle : \mathcal{H}_N(\mathbb{C}^2) \times \mathcal{H}_N(\mathbb{C}^2) \rightarrow \mathbb{F}$  such that

$$\langle [\vec{Y}], [\vec{Y}'] \rangle = \delta_{\vec{Y}, \vec{Y}'} \text{Eu}(T_{\vec{Y}} \mathfrak{M}_{N,n}(\mathbb{C}^2))^{-1} \quad (3.1)$$



on  $\mathcal{H}_{N,n}(\mathbb{C}^2) \times \mathcal{H}_{N,n}(\mathbb{C}^2)$ . By the localization theorem, it thus follows that the collection of vectors  $[\vec{Y}]$  for all  $n \geq 0$  forms an orthogonal  $\mathbb{F}$ -basis for  $\mathcal{H}_N(\mathbb{C}^2)$ . In particular, the equivariant fundamental class of the moduli space  $\mathfrak{M}_{N,n}(\mathbb{C}^2)$  in  $H_{\mathbb{T}}^{\bullet}(\mathfrak{M}_{N,n}(\mathbb{C}^2))$  is given by  $[\mathfrak{M}_{N,n}(\mathbb{C}^2)] = \sum_{\vec{Y}} [\vec{Y}]$ .

**Theorem 3.2** (a) *The vector space  $\mathcal{H}_N(\mathbb{C}^2)$  is the Verma module for the  $\mathcal{W}(\mathfrak{gl}_N)$ -algebra of central charge*

$$c = N + (N-1)N(N+1)\varepsilon^2 \quad \text{with} \quad \varepsilon^2 := \frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2}$$

and highest weight  $\vec{\lambda} = \frac{\vec{a}}{\sqrt{\epsilon_1 \epsilon_2}} + \varepsilon \vec{\rho}$  where  $\vec{\rho}$  is the Weyl vector of  $\mathfrak{gl}_N$ .

(b) *The vector*

$$\psi_{\mathbb{C}^2} = \sum_{n=0}^{\infty} [\mathfrak{M}_{N,n}(\mathbb{C}^2)]$$

in the completion  $\prod_{n \geq 0} \mathcal{H}_{N,n}(\mathbb{C}^2)$  is a Whittaker vector for this highest weight representation.

Part (a) of this theorem generalizes the constructions of Nakajima [79, 80] and Vasserot [107] of Fock space representations, using geometric Hecke correspondences on the (equivariant) cohomology of Hilbert schemes  $\text{Hilb}^n(\mathbb{C}^2)$ , of the Heisenberg algebra  $\mathcal{A}_1(\mathbb{C}^2) = \mathfrak{h} = \widehat{\mathfrak{gl}}_1$  in the rank one case  $N = 1$ ; this is the vertex algebra underlying the  $c = 1$  free boson conformal field theory. For  $N > 1$  the theorem was proved independently by Schiffmann–Vasserot [93] and by Maulik–Okounkov [73]. The vertex algebra  $\mathcal{A}_N(\mathbb{C}^2) = \mathfrak{h} \oplus \mathcal{W}(\mathfrak{sl}_N)$  underlies the  $A_{N-1}$  Toda conformal field theories; for  $N = 2$  the  $\mathcal{W}(\mathfrak{sl}_2)$ -algebra is the Virasoro algebra underlying Liouville theory. The technical difficulties involved in the proofs is the lack of presentation of generic  $\mathcal{W}(\mathfrak{sl}_N)$ -algebras in terms of generators and relations (see §C), which are overcome by realising the induced action of the vertex algebra  $\mathcal{W}(\mathfrak{gl}_N)$  via its embedding into a larger infinite-dimensional Hopf algebra. The Whittaker vector  $\psi_{\mathbb{C}^2}$  is sometimes also called a *Gaiotto state* [49]; it plays the role of a kind of “coherent state” for the geometric highest weight representation.

This theorem is regarded as providing a rigorous proof of the AGT conjectures [3, 112, 2] for pure  $N = 2$  gauge theory on  $\mathbb{C}^2$  in the following sense. Define an operator  $\mathbf{q}^{L_0}$  on  $\mathcal{H}_N(\mathbb{C}^2)$  by letting it act as scalar multiplication by  $\mathbf{q}^n$  on each graded component  $\mathcal{H}_{N,n}(\mathbb{C}^2)$  for  $n \geq 0$ . It then follows immediately from the projection formula (3.1) that Nekrasov’s partition function (2.2) for  $N = 2$  gauge theory on  $\mathbb{C}^2$  takes the form of a conformal block

$$\mathcal{Z}_{\mathbb{C}^2}^{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; \mathbf{q}) = \langle \psi_{\mathbb{C}^2}, \mathbf{q}^{L_0} \psi_{\mathbb{C}^2} \rangle.$$

Via term by term matching of explicit expressions, the partition functions of  $N = 2$  quiver gauge theories on  $\mathbb{C}^2$  have been found to agree with the conformal blocks of  $A_{N-1}$  Toda field theories [3, 112]. A rigorous proof of this correspondence requires a geometric construction of chiral vertex operators whose suitable matrix elements on  $\mathcal{H}_N(\mathbb{C}^2)$  coincide with the Nekrasov partition functions. It is always possible to define such operators in terms of the Euler classes of Carlsson–Okounkov type Ext-bundles [27] which are defined using the universal sheaf on  $\mathfrak{M}_{N,n}(\mathbb{C}^2) \times \mathbb{P}^2$ ; they have completely factorised matrix elements in the fixed point basis  $[\vec{Y}]$  of  $\mathcal{H}_N(\mathbb{C}^2)$ . However, a characterization of these operators as primary fields of the  $\mathcal{W}(\mathfrak{gl}_N)$ -algebra is currently not known in generality. For  $N = 1$ , this construction produces vertex operators which are primary fields of the Heisenberg algebra  $\mathfrak{h}$  in terms of bosonic exponentials of Nakajima operators [27]. Hence the AGT conjecture is completely proven in the rank one case. For  $N = 2$  it was shown by [43, 55] that these matrix elements correspond to primary field insertions in conformal blocks. One of the main interests in the AGT correspondence from the perspective of conformal field theory is in fact



the existence of the distinguished basis  $[\vec{Y}]$  for the Verma module which has no natural origin from the purely two-dimensional point of view.

For the pure  $\mathcal{N} = 2$  gauge theory on  $X = \mathbb{C}^2$ , the instanton part of the Seiberg–Witten prepotential is given by [83, 82, 85]

$$\mathcal{F}_{\mathbb{C}^2}^{\text{inst}}(\vec{a}; \mathbf{q}) = \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} -\epsilon_1 \epsilon_2 \log \mathcal{Z}_{\mathbb{C}^2}^{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; \mathbf{q}) ,$$

where the normalization is the equivariant volume of  $X = \mathbb{C}^2$ , i.e.  $\int_X 1 = \frac{1}{\epsilon_1 \epsilon_2}$ . The Donagi–Witten integrable system in this case is the twisted Toda integrable system of type  $A_{N-1}$ , which is a Hitchin system on the Riemann sphere  $C = \mathbb{P}^1$  with two irregular punctures at  $z = 0, \infty$ . The Whittaker vector  $\psi_{\mathbb{C}^2}$  is a state in the representation corresponding to an irregular puncture; “irregular” here refers to wild singularities of the Seiberg–Witten spectral curve  $\Sigma$  corresponding to non-conformal behaviour of the conformal blocks in this case.

In the  $\mathcal{N} = 2$  gauge theory, the vector multiplet scalars  $\mathbf{I}_p = \text{Tr } \phi^p$  for all  $p \geq 1$  give an infinite of commuting Hamiltonians underlying the Donagi–Witten integrable system. The quantization of this integrable system provided by the  $\Omega$ -deformation is realised by the action of an infinite-dimensional commutative algebra of integrals of motion acting in the representation  $\mathcal{H}_N(\mathbb{C}^2)$ , which are diagonalised in the fixed point basis  $[\vec{Y}]$ . They are realised *geometrically* on  $\mathcal{H}_N(\mathbb{C}^2)$  as cup multiplication by the Chern classes  $\mathbf{I}_p = c_{p-1}(\mathbf{V}^N)$  of the *natural vector bundle*  $\mathbf{V}^N$  on  $\mathfrak{M}_N(\mathbb{C}^2)$  [95]; they can be read off from the  $\tilde{\mathbb{T}}$ -equivariant Chern character

$$\text{Ch}(\mathbf{V}_{\vec{Y}}^N) = \sum_{l=1}^r \sum_{s \in Y_l} e^{-L^t(s) \epsilon_1 - A^t(s) \epsilon_2}$$

at the fixed point  $[\vec{Y}]$ . In the rank one case  $N = 1$ , they can be identified with Nakajima operators which span a commutative subalgebra of the universal enveloping algebra of the Heisenberg algebra  $\mathfrak{h}$ ; in particular, this identifies  $\mathbf{I}_1$  with the energy operator  $L_0$  and  $\mathbf{I}_2$  with the bosonised Hamiltonian of a quantum trigonometric Calogero–Sutherland model with infinitely many particles and coupling constant  $\beta^{-1}$ . In fact, a stronger statement is true for any  $N \geq 1$ : With  $\Lambda_{\mathbb{F}} := \Lambda \otimes_{\mathbb{C}} \mathbb{F}$  the algebra of symmetric functions over the field  $\mathbb{F}$  (see §B), there is an isomorphism  $\mathcal{H}_N(\mathbb{C}^2) \cong \Lambda_{\mathbb{F}}^{\otimes N}$ , as representations of the  $\mathcal{W}(\mathfrak{gl}_N)$ -algebra, which sends

$$[\vec{Y}] \longmapsto J_{\vec{Y}} .$$

Here  $J_{\vec{Y}}$  are the generalized Jack symmetric functions which are the eigenfunctions of a quantum deformation of the Calogero–Moser–Sutherland integrable system [1, 75, 96]; for  $N = 1$  they coincide with the usual Jack functions  $J_Y(x; \beta)$ , see §B.3.

### 3.3 AGT duality on $[\mathbb{C}^2/\mathbb{Z}_k]$

Next we consider the quotient stack  $X = [\mathbb{C}^2/\mathbb{Z}_k]$ . Then the  $\tilde{\mathbb{T}}$ -equivariant cohomology  $\mathcal{H}_{\vec{w}}([\mathbb{C}^2/\mathbb{Z}_k])$  of the corresponding quiver variety  $\mathfrak{M}_{\xi_0}(\vec{w}) := \bigsqcup_{\vec{v} \in \mathbb{Z}_{\geq 0}^k} \mathfrak{M}_{\xi_0}(\vec{v}, \vec{w})$  for  $\xi_0 \in \mathbf{C}_0$  conjecturally carries a representation of the coset construction [12, 90, 9, 113]

$$\mathcal{A}_N([\mathbb{C}^2/\mathbb{Z}_k]) := \frac{(\widehat{\mathfrak{gl}}_{\kappa})_N}{(\widehat{\mathfrak{gl}}_{\kappa-k})_N} \cong \mathfrak{h} \oplus (\widehat{\mathfrak{sl}}_k)_N \oplus \frac{(\widehat{\mathfrak{sl}}_N)_k \oplus (\widehat{\mathfrak{sl}}_N)_{\kappa-k}}{(\widehat{\mathfrak{sl}}_N)_{\kappa}}$$

where  $\kappa \in \mathbb{C}$  is given in terms of the equivariant parameters  $(\epsilon_1, \epsilon_2, \vec{a})$  and the isomorphism is a consequence of level-rank duality. This conjecture generalises Nakajima’s seminal construction [78]

of level  $N$  representations of the affine Lie algebra  $\widehat{\mathfrak{gl}}_k$  on the cohomology of  $\widehat{A}_{k-1}$  quiver varieties  $\mathfrak{M}_\xi(\vec{v}, \vec{w})$  via geometric Hecke correspondences. A gauge theory realisation of Nakajima's construction was provided by Vafa and Witten [106] in the framework of a topological twisting of  $\mathcal{N} = 4$  gauge theory in four dimensions, which is defined as the dimensional reduction of ten-dimensional  $\mathcal{N} = 1$  supersymmetric Yang–Mills theory on the trivial complex vector bundle of rank three in the limit where the fiber collapses to a point. They showed that the corresponding partition functions, which are generating functions for the Euler characteristics of instanton moduli spaces, compute characters of highest weight representations of  $\widehat{\mathfrak{gl}}_k$  at level  $N$  (see also [37]); a rigorous derivation, based on realising the instanton moduli space as the quiver variety  $\mathfrak{M}_{\xi_0}(\vec{w})$ , is given in [47].

For  $N = 1$ , a vertex algebra realization of the  $j$ -th fundamental representation of  $\mathcal{A}_1([\mathbb{C}^2/\mathbb{Z}_k]) = \mathfrak{h} \oplus (\widehat{\mathfrak{sl}}_k)_1$  on the equivariant cohomology of the quiver variety  $\mathfrak{M}_{\xi_0}(\vec{w}_j)$  is given in [77]; this is the chiral algebra underlying the  $U(k)$  WZW conformal field theory at level one. For  $N = 2$ , the algebra  $\mathcal{A}_2([\mathbb{C}^2/\mathbb{Z}_k])$  is the sum of  $(\widehat{\mathfrak{gl}}_k)_2$  and the  $\mathcal{N} = 1$  supersymmetric extension of the Virasoro algebra; for  $k = 2$  this is the vertex algebra underlying  $\mathcal{N} = 1$  supersymmetric Liouville theory. For general  $N$  and  $k$ , the pertinent two-dimensional conformal field theory is the  $\mathbb{Z}_k$ -parafermionic Toda field theory of type  $A_{N-1}$ . Beyond these results and some explicit checks (see e.g. [59, 4, 104]), not much is rigorously proven at present for the AGT conjecture on the quotient stack  $[\mathbb{C}^2/\mathbb{Z}_k]$ . One line of attack has been to take the Uglov limit of the related five-dimensional gauge theory discussed in §2.7, wherein a  $q$ -deformed vertex algebra conjecturally acts on the  $\widetilde{T}$ -equivariant K-theory of the instanton moduli space  $\mathfrak{M}_N(\mathbb{C}^2)$  [8]; this algebra is called the elliptic Hall algebra (among various other names). It was conjectured by [10] that the Uglov limit of the level  $N$  representation of the elliptic Hall algebra tends to the conformal algebra  $\mathcal{A}_N([\mathbb{C}^2/\mathbb{Z}_k])$  preserving the special fixed point bases where the vertex operators have completely factorised forms; checks of this proposal can be found in e.g. [62, 98]. In the rank one case  $N = 1$ , the geometric action of the elliptic Hall algebra is constructed using Hecke correspondences in [92] and vertex operators in [44], while the K-theory versions of the vertex operators geometrically defined using Ext-bundles that compute matrix elements in  $q$ -deformed conformal field theory are constructed in [28] and shown to have a bosonic exponential form in terms of deformed Heisenberg operators as in [8].

We can also connect this duality to quantum integrability. For  $N = 1$ , there are isomorphisms  $\mathcal{H}_{\vec{w}_j}([\mathbb{C}^2/\mathbb{Z}_k]) \cong \Lambda_{\mathbb{F}}$  for each  $j = 0, 1, \dots, k-1$ , as level one representations of the affine algebra  $\widehat{\mathfrak{gl}}_k$ , which sends

$$[Y] \longmapsto U_Y(x; \beta, k) .$$

Here  $U_Y(x; \beta, k)$  are the Uglov symmetric functions of rank  $k$  associated with the coloured Young diagram  $Y$  (see §B.3), which are the eigenfunctions of a spin generalization of the Calogero–Sutherland model [105]. The first few integrals of motion are given in [10].

### 3.4 AGT duality on $X_k$

We finally turn to the minimal resolution  $X = X_k$ . As a first check, let us compute the Vafa–Witten partition function in the chamber  $\mathcal{C}_\infty$  which is the generating function

$$z_{X_k}^{\text{VW}}(\mathbf{q}, \vec{\xi})_{\vec{w}} := \sum_{\vec{u} \in \mathcal{U}_{\vec{w}}} \vec{\xi}^{\vec{u}} \sum_{\Delta \in \frac{1}{2Nk} \mathbb{Z}} \mathbf{q}^{\Delta + \frac{1}{2N} \vec{u} \cdot C^{-1} \vec{u}} \int_{\mathfrak{M}_{\vec{u}, \Delta, \vec{w}}(X_k)} \text{Eu}(T\mathfrak{M}_{\vec{u}, \Delta, \vec{w}}(X_k))$$

for the Euler characteristic of the moduli space  $\mathfrak{M}_{\vec{w}}(X_k) := \bigsqcup_{\vec{u}, \Delta} \mathfrak{M}_{\vec{u}, \Delta, \vec{w}}(X_k)$ . Applying the localization theorem, the Euler classes cancel between the denominator and numerator, and the resulting

expression simply enumerates fixed points  $(\vec{Y}, \vec{u})$ . Analogously to the rank one computation of §2.5, the weighted combinatorial sum gives [26]

$$\mathcal{Z}_{X_k}^{\text{VW}}(\mathbf{q}, \vec{\xi})_{\vec{w}} = \prod_{j=0}^{k-1} \left( \frac{\chi^{\hat{\omega}_j}(\mathbf{q}, \vec{\xi})}{\hat{\eta}(\mathbf{q})} \right)^{w_j} \quad (3.3)$$

which is the character of the affine Lie algebra  $\hat{\mathfrak{gl}}_k$  at level  $N$  expected from Nakajima's highest weight representations on the cohomology of quiver varieties  $\mathfrak{M}_{\xi}(\vec{v}, \vec{w})$ . The formula (3.3) formally includes the case  $k = 1$ , where it reduces to the Vafa–Witten partition function

$$\mathcal{Z}_{\mathbb{C}^2}^{\text{VW}}(\mathbf{q})_N = \hat{\eta}(\mathbf{q})^{-N} \quad (3.4)$$

for  $N = 4$   $U(N)$  gauge theory on  $\mathbb{C}^2$ .

This result also confirms the  $SL(2, \mathbb{Z})$  modularity (S-duality) of the  $N = 4$  gauge theory partition function, which has a natural explanation in the  $(2, 0)$  theory on the product Riemannian six-manifold

$$M = X_k \times T^2,$$

where the real torus  $T^2$  has complex structure modulus  $\tau$ , interpreted as the complexified gauge coupling constant, and local coordinates  $(x, y) \in \mathbb{R}^2$ . For instance, in the abelian case  $N = 1$  the  $(2, 0)$  theory governs a self-dual closed real three-form  $H \in \Omega_{\mathbb{Z}}^3(M)$  with integral periods which is the curvature of an abelian gerbe with connection. It can be parameterized as

$$H = F \wedge dx + *F \wedge dy,$$

where  $F \in \Omega_{\mathbb{Z}}^2(X_k)$  is regarded as the curvature two-form of a complex line bundle over  $X_k$  with connection. The limit in which the torus  $T^2$  collapses to a point thereby sends the six-dimensional abelian gerbe theory on  $M$  to four-dimensional  $U(1)$  gauge theory on  $X_k$ . The S-duality symmetry of the quantum gauge theory is induced geometrically in the quantum gerbe theory by the automorphism group of  $T^2$  which acts as  $SL(2, \mathbb{Z})$  modular transformations of  $\tau$ . For the higher rank case  $N > 1$ , we first take the limit where the torus  $T^2$  collapses to a circle to describe the non-abelian gerbe theory as five-dimensional  $N = 1$   $U(N)$  gauge theory on the product  $X_k \times S^1$ , and then further collapse the remaining circle  $S^1$  to get the  $U(N)$  Vafa–Witten theory on  $X_k$ ; see [108] for further details and explanations.

Using blowup formulas, for  $N = 2$  the partition functions of  $N = 2$  gauge theories on  $X_2$  have been matched with conformal blocks of  $N = 1$  supersymmetric Liouville theory [17, 18], while a representation theoretic interpretation of the blowup equations in this case has been provided in terms of vertex operator algebras and conformal blocks by [15, 14]. But beyond this very little is known about the AGT correspondence in generality for Nakajima quiver varieties with stability parameters from the chamber  $\mathcal{C}_{\infty}$ . For  $N = 1$  we can say a bit more. We note first of all that the localized equivariant cohomology of the moduli spaces  $\mathfrak{M}_{\vec{w}_j}(X_k)$  can be decomposed as

$$\mathcal{H}_{\vec{w}_j}(X_k) = \bigoplus_{\vec{u} \in \mathcal{U}_{\vec{w}_j}} \mathcal{H}_{\vec{u}, \vec{w}_j}(X_k) \quad \text{with} \quad \mathcal{H}_{\vec{u}, \vec{w}_j}(X_k) \cong \bigoplus_{n=0}^{\infty} H_{\mathbb{T}}^{\bullet}(\text{Hilb}^n(X_k))_{\text{loc}} \otimes \mathbb{C}(\epsilon_1, \epsilon_2)[Q + \omega_j].$$

**Theorem 3.5** (a) The vector space  $\mathcal{H}_{\vec{w}_j}(X_k)$  is the  $j$ -th fundamental representation of  $\hat{\mathfrak{gl}}_k$  at level one with weight spaces  $\mathcal{H}_{\vec{u}, \vec{w}_j}(X_k)$  of highest weight  $\gamma_{\vec{u}} + \omega_j$ .

(b) The vector space  $\mathcal{H}_{\vec{u}, \vec{w}_j}(X_k)$  is the Verma module for the Virasoro algebra with central charge

$c = k$  and conformal dimension  $\Delta_{\vec{u}} = \frac{1}{2} \vec{u} \cdot C^{-1} \vec{u}$ .

(c) The vector

$$\psi_{X_k} = \sum_{n=0}^{\infty} \sum_{\vec{u} \in \mathcal{U}_{\vec{w}_j}} [\mathfrak{M}_{\vec{u}, n, \vec{w}_j}(X_k)]$$

in the completion  $\prod_{n, \vec{u}} \mathcal{H}_{\vec{u}, n, \vec{w}_j}(X_k)$  is a Whittaker vector for this highest weight representation.

This geometric action of  $\mathcal{A}_1([\mathbb{C}^2/\mathbb{Z}_k]) = (\widehat{\mathfrak{gl}}_k)_1$  is constructed in [91] by using Nakajima correspondences for the affine algebra  $\mathfrak{h} \oplus \mathfrak{h}_{\mathbb{Q}}$  on the equivariant cohomology of the Hilbert schemes  $\text{Hilb}^n(X_k)$ , and then applying the Frenkel–Kac construction to the Heisenberg algebra  $\mathfrak{h}_{\mathbb{Q}}$  associated to the  $A_{k-1}$  root lattice  $\mathbb{Q}$ . At present, however, the geometric relationship between the level one  $\widehat{\mathfrak{gl}}_k$ -modules  $\mathcal{H}_{\vec{w}_j}([\mathbb{C}^2/\mathbb{Z}_k])$  and  $\mathcal{H}_{\vec{w}_j}(X_k)$  is not understood.

The underlying Seiberg–Witten geometry is found by computing the limit [26]

$$\lim_{\epsilon_1, \epsilon_2 \rightarrow 0} -k \epsilon_1 \epsilon_2 \log \mathcal{Z}_{X_k}^{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; \mathbf{q}, \vec{\xi})_{\vec{w}} = \frac{1}{k} \mathcal{F}_{\mathbb{C}^2}^{\text{inst}}(\vec{a}; \mathbf{q}) .$$

This result shows that the  $\Omega$ -deformed  $\mathcal{N} = 2$  gauge theory on  $X_k$  is a quantization of the same  $U(N)$ -Hitchin system on a two-punctured sphere  $C = \mathbb{P}^1$  (the  $A_{N-1}$ -Toda system) as for the pure  $\mathcal{N} = 2$  gauge theory on  $\mathbb{C}^2$ . For  $N = 1$  we can again elucidate further the structure of this quantum integrable system. With  $\mathbf{V}_{\vec{w}_j}$  denoting the natural vector bundle on the moduli space  $\mathfrak{M}_{\vec{w}_j}(X_k)$ , one can again show that the equivariant Chern classes  $\mathbf{I}_p = c_{p-1}(\mathbf{V}_{\vec{w}_j})$  with  $p \geq 1$  form an infinite system of commuting multiplication operators which are diagonalized in fixed point basis  $[\vec{Y}, \vec{u}]$  of  $\mathcal{H}_{\vec{w}_j}(X_k)$  [91]; in particular,  $\mathbf{I}_1$  can be identified with the Virasoro operator  $L_0$  for  $\widehat{\mathfrak{gl}}_k$  and  $\mathbf{I}_2$  as the sum of  $k$  non-interacting quantum Calogero–Sutherland Hamiltonians with prescribed couplings  $\beta^{(i)-1} = -\epsilon_2^{(i)}/\epsilon_1^{(i)}$  for  $i = 1, \dots, k$ . In fact, in this case there is an isomorphism

$$\mathcal{H}_{\vec{w}_j}(X_k) \cong \Lambda_{\mathbb{F}}^{\otimes k} \otimes \mathbb{C}(\epsilon_1, \epsilon_2)[\mathbb{Q} + \omega_j] ,$$

as representations of the affine algebra  $\widehat{\mathfrak{gl}}_k$ , which sends

$$[\vec{Y}, \vec{u}] \longrightarrow J_{Y^1}(x; \beta^{(1)}) \otimes \cdots \otimes J_{Y^k}(x; \beta^{(k)}) \otimes (\gamma_{\vec{u}} + \omega_j) .$$

Understanding the relations between the two level one representations of  $\widehat{\mathfrak{gl}}_k$  provided by the chambers  $\mathbf{C}_0$  and  $\mathbf{C}_{\infty}$  at the level of symmetric functions and their combinatorics would serve as a good step towards a clearer understanding of the relationships between the AGT dualities, as well as a closer step towards their proofs. This would also entail unveiling the relations between the different fixed point bases of  $\mathcal{H}_{\vec{w}_j}([\mathbb{C}^2/\mathbb{Z}_k])$  and  $\mathcal{H}_{\vec{w}_j}(X_k)$ , as well as the corresponding quantum integrable systems.

#### 4 $\mathcal{N} = 2$ gauge theories in six dimensions

The purpose of this section is to compute and understand the equivariant partition functions of the topologically twisted maximally supersymmetric Yang–Mills theory in six dimensions in parallel to our four-dimensional  $\mathcal{N} = 2$  gauge theories. In the Coulomb branch of the  $U(N)$  gauge theory on  $\mathbb{C}^3$ , this has been computed in [31] using equivariant localization techniques in the case when the complex equivariant parameters  $\vec{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3)$  of the natural scaling action of the three-dimensional torus  $\mathbb{T}^3$  on the affine space  $\mathbb{C}^3$  satisfy the Calabi–Yau constraint  $\epsilon = 0$  where

$$\epsilon := \epsilon_1 + \epsilon_2 + \epsilon_3 ,$$

i.e. the action of  $\mathbb{T}^3 \subset SL(3, \mathbb{C})$  is reduced to an action of a two-dimensional torus. This defines the “Calabi–Yau specialization” of the  $\Omega$ -deformation, as it is the condition that the  $\mathbb{T}^3$ -action preserve the holomorphic three-form of  $\mathbb{C}^3$ , and it is the six-dimensional analog of the anti-diagonal torus action which is commonly used in four dimensions; there it has various physical interpretations, e.g. as the condition for self-duality of the graviphoton background in four-dimensional  $\mathcal{N} = 2$  supergravity, or in the context of the AGT duality as the condition for vanishing background  $U(1)$  charge  $\varepsilon = 0$  in the dual conformal field theory. Here we extend this computation to arbitrary torus actions, and then describe some intriguing four-dimensional reductions of the six-dimensional gauge theory.

#### 4.1 Instanton partition function

The  $\mathcal{N} = 2$  gauge theory of interest can be defined on any complex Kähler threefold  $(X, \omega)$  by dimensional reduction of ten-dimensional  $\mathcal{N} = 1$  supersymmetric Yang–Mills theory on the trivial complex vector bundle  $X \times \mathbb{C}^2$  of rank two over  $X$  in the limit where the fiber collapses to a point. Restricting for simplicity to the case that  $X$  is Calabi–Yau with no compact divisors, supersymmetric localization identifies the relevant moduli problem as that associated to solutions of the  $\delta$ -fixed point equations [58, 31]

$$F^{1,1} \wedge \omega \wedge \omega = 0 \quad \text{and} \quad \nabla_A \phi = 0$$

for an integrable connection  $A$  on a holomorphic vector bundle over  $X$  and a local section  $\phi$  of its adjoint bundle. In analogy with the four-dimensional case we shall refer to solutions of these equations as (generalised) instantons. In some instances this moduli problem is associated to instantons on a noncommutative deformation of  $X$ , for example in Type IIA string theory when it describes bound states of D0-branes and D2-branes in D6-branes [110]. Depending on the choice of stability parameters, the instanton moduli space can also be parameterized as a moduli space of torsion-free sheaves on  $X$  [58, 31, 32]; in the rank one case this gauge theory is identified by [86] with the K-theoretic Donaldson–Thomas theory of  $X$ .

In this section we will work primarily in the Coulomb branch of  $U(N)$  gauge theory on  $X = \mathbb{C}^3$ . The  $\Omega$ -deformed gauge theory is defined by realising it as the worldvolume theory of  $N$  separated D6-branes in Type IIA string theory and lifting it to its dual M-theory description on a  $\mathbb{T}N_N \times \mathbb{C}^3$ -bundle  $\mathcal{M}_\varrho$  over a circle  $S^1$  whose radius  $\varrho$  is identified with the string coupling constant, where  $\mathbb{T}N_N$  is the hyper-Kähler  $N$ -centered Taub–NUT space which is a local  $S^1$ -fibration over  $\mathbb{R}^3$  (for the present discussion we could also replace the generic  $S^1$  fibre with  $\mathbb{R}$ , which gives an ALE space  $X_N$ ). The total space of  $\mathcal{M}_\varrho$  is the quotient of  $\mathbb{T}N_N \times \mathbb{C}^3 \times \mathbb{R}$  by the  $\mathbb{Z}$ -action

$$n \triangleright (p, z_1, z_2, z_3, x) = (g^n(p), t_1^n z_1, t_2^n z_2, t_3^n z_3, x + 2\pi n \varrho) ,$$

where  $p \in \mathbb{T}N_N$ ,  $(z_1, z_2, z_3) \in \mathbb{C}^3$ ,  $x \in \mathbb{R}$ ,  $g$  is an isometry of  $\mathbb{T}N_N$  and  $(t_1, t_2, t_3) \in \mathbb{T}^3$  is an element of the maximal torus of the complex rotation group  $GL(3, \mathbb{C})$  with  $t_i := e^{\epsilon_i}$ , while  $n \in \mathbb{Z}$  is the D0-brane (instanton) charge of the dual Type IIA string theory description in ten dimensions. Taking the collapsing limit  $\varrho \rightarrow 0$  then gives the desired  $\Omega$ -deformation.

The instanton partition function for  $\mathcal{N} = 2$  gauge theory on  $\mathbb{C}^3$  is then given by

$$\mathcal{Z}_{\mathbb{C}^3}^{\text{inst}}(\vec{\epsilon}, \vec{a}; \mathbf{q}) = \sum_{n=0}^{\infty} \mathbf{q}^n \int_{[\mathfrak{M}_{N,n}(\mathbb{C}^3)]^{\text{vir}}} 1 , \quad (4.1)$$

where in this case  $\mathbf{q} = e^{-\varrho}$ . Here  $[\mathfrak{M}_{N,n}(\mathbb{C}^3)]^{\text{vir}}$  is the virtual fundamental class of the moduli scheme  $\mathfrak{M}_{N,n}(\mathbb{C}^3)$  of  $U(1)^N$  (generalised)  $n$ -instanton solutions of the six-dimensional noncommutative  $\mathcal{N} = 2$  gauge theory, or equivalently a particular moduli space of torsion-free sheaves  $\mathcal{E}$  on

$\mathbb{P}^3 = \mathbb{C}^3 \cup \varphi_\infty$  of rank  $N$  with  $\text{ch}_3(\mathcal{E}) = -n$  which are framed on a plane  $\varphi_\infty \cong \mathbb{P}^2$  at infinity [32], and  $\vec{a} = (a_1, \dots, a_N)$  is the vector of Higgs vevs which are the parameters of the induced  $(\mathbb{C}^\times)^N$ -action on the moduli space. The moduli spaces  $\mathfrak{M}_{N,n}(\mathbb{C}^3)$  have generic complex dimension  $3N - n$ , and here we will treat them in the framework of perfect obstruction theory, as they are generally not smooth varieties. We will understand the integral in (4.1) as a  $\mathbb{T}$ -equivariant integral, where  $\mathbb{T} = \mathbb{T}^3 \times (\mathbb{C}^\times)^N$ , and define it properly via the localization theorem; it arises formally from localizing the path integral to  $\int_{\mathfrak{M}_{N,n}(\mathbb{C}^3)} \text{Eu}(N\mathfrak{M}_{N,n}(\mathbb{C}^3))$ , where  $\text{Eu}(N\mathfrak{M}_{N,n}(\mathbb{C}^3))$  is the equivariant Euler class of the obstruction bundle on the instanton moduli space which arises from integration of the quartic fermion terms in the action of the supersymmetric gauge theory. Since  $\text{Eu}(N\mathfrak{M}_{N,n}(\mathbb{C}^3)) \in CH_{\mathbb{T}}^\bullet(\mathfrak{M}_{N,n}(\mathbb{C}^3))$ , it takes values in the coefficient ring  $CH_{\mathbb{T}}^\bullet(\text{pt}) = \mathbb{C}[\epsilon_1, \epsilon_2, \epsilon_3, a_1, \dots, a_N]_{(\vec{\epsilon}, \vec{a})}$  for the equivariant Chow theory of the moduli space, which is the localization of the ring  $\mathbb{C}[\epsilon_1, \epsilon_2, \epsilon_3, a_1, \dots, a_N]$  at the maximal ideal  $(\vec{\epsilon}, \vec{a})$  generated by  $\epsilon_1, \epsilon_2, \epsilon_3, a_1, \dots, a_N$ .

The  $\mathbb{T}$ -fixed points of the instanton moduli space  $\mathfrak{M}_{N,n}(\mathbb{C}^3)$  are isolated and parameterized by  $N$ -vectors of three-dimensional Young diagrams (plane partitions)  $\vec{\pi} = (\pi_1, \dots, \pi_N)$  of total size  $|\vec{\pi}| = \sum_l |\pi_l| = n$  (see §A.3). Each component partition  $\pi_l$  specifies a  $\mathbb{T}^3$ -invariant  $U(1)$  noncommutative instanton of topological charge  $|\pi_l|$ , or equivalently a monomial ideal  $I \subset \mathbb{C}[z_1, z_2, z_3] = \mathcal{O}_{\mathbb{C}^3}$  of codimension  $|\pi_l|$ . Applying the virtual localization theorem in equivariant Chow theory to the integral in (4.1) yields the combinatorial expansion

$$\int_{[\mathfrak{M}_{N,n}(\mathbb{C}^3)]^{\text{vir}}} 1 = \sum_{|\vec{\pi}|=n} \frac{\text{Eu}(N_{\vec{\pi}}\mathfrak{M}_{N,n}(\mathbb{C}^3))}{\text{Eu}(T_{\vec{\pi}}\mathfrak{M}_{N,n}(\mathbb{C}^3))}. \quad (4.2)$$

In contrast to the maximally supersymmetric gauge theory in four dimensions (the Vafa–Witten theory), here the ratios of Euler classes do not cancel in general, as the tangent and obstruction bundles do not necessarily coincide (though they have the same rank). Hence the partition function is generally a complicated function of the equivariant parameters and the Higgs field vevs. In fact, the situation closely resembles the case of pure  $\mathcal{N} = 2$  gauge theory in four dimensions, except that we work with the virtual tangent bundle  $T^{\text{vir}}\mathfrak{M}_{N,n}(\mathbb{C}^3) = T\mathfrak{M}_{N,n}(\mathbb{C}^3) \ominus N\mathfrak{M}_{N,n}(\mathbb{C}^3)$  rather than the stable tangent bundle  $T\mathfrak{M}_{N,n}(\mathbb{C}^3)$  (which is generally not well-defined here); in particular, the equivariant characteristic classes in our case involve rational functions of the equivariant parameters rather than polynomials. Further details concerning such equivariant integrals can be found in [101, §3.5].

To compute the virtual equivariant characteristic classes in (4.2), we use the local model of the instanton moduli space developed in [31] from the instanton quantum mechanics for  $n$  D0-branes inside  $N$  D6-branes on  $\mathbb{C}^3$ . There it was shown that these classes can be computed from the character of the equivariant instanton deformation complex

$$\begin{array}{ccccc} & & \text{End}_{\mathbb{C}}(V_{\vec{\pi}}) \otimes Q & & \\ & & \oplus & & \\ \text{End}_{\mathbb{C}}(V_{\vec{\pi}}) & \longrightarrow & \text{Hom}_{\mathbb{C}}(W_{\vec{\pi}}, V_{\vec{\pi}}) & \longrightarrow & \text{End}_{\mathbb{C}}(V_{\vec{\pi}}) \otimes \wedge^2 Q \\ & & \oplus & & \oplus \\ & & \text{End}_{\mathbb{C}}(V_{\vec{\pi}}) \otimes \wedge^3 Q & & \text{Hom}_{\mathbb{C}}(V_{\vec{\pi}}, W_{\vec{\pi}}) \otimes \wedge^3 Q \end{array} \quad (4.3)$$

where  $Q \cong \mathbb{C}^3$  is the fundamental representation of  $\mathbb{T}^3$  with weight  $(1, 1, 1)$ , while  $V_{\vec{\pi}}$  and  $W_{\vec{\pi}}$  are complex vector spaces of dimensions  $n$  and  $N$  parameterizing the D0-branes and D6-brane gauge degrees of freedom, respectively. They admit the decompositions

$$V_{\vec{\pi}} = \sum_{l=1}^N e_l \sum_{(i,j,k) \in \pi_l} t_1^{i-1} t_2^{j-1} t_3^{k-1} \quad \text{and} \quad W_{\vec{\pi}} = \sum_{l=1}^N e_l \quad (4.4)$$



as  $\tilde{\mathbb{T}}$  representations regarded as polynomials in  $t_i$ ,  $i = 1, 2, 3$ , and  $e_l := e^{a_l}$ ,  $l = 1, \dots, N$ , i.e. as elements in the representation ring of  $\tilde{\mathbb{T}}$ . The first cohomology of this complex is a local model of the tangent space to the moduli space at the fixed point  $\tilde{\pi}$ , while the second cohomology parameterizes obstructions. Its equivariant Chern character is easily computed as

$$\begin{aligned} \text{Ch}(T_{\tilde{\pi}}^{\text{vir}} \mathfrak{M}_{N,n}(\mathbb{C}^3)) &= W_{\tilde{\pi}}^* \otimes V_{\tilde{\pi}} - (t_1 t_2 t_3)^{-1} V_{\tilde{\pi}}^* \otimes W_{\tilde{\pi}} \\ &\quad + (t_1^{-1} - 1) (t_2^{-1} - 1) (t_3^{-1} - 1) V_{\tilde{\pi}}^* \otimes V_{\tilde{\pi}}, \end{aligned} \quad (4.5)$$

and the inverse of the corresponding top Chern polynomial yields the desired ratio in (4.2); here the dual involution acts on the weights as  $t_i^* = t_i^{-1}$  and  $e_l^* = e_l^{-1}$ .

In [31] it is shown that at the Calabi–Yau specialization  $\epsilon = 0$  of the  $\Omega$ -deformation, the Euler classes in (4.2) coincide up to a sign,

$$\text{Eu}(N_{\tilde{\pi}} \mathfrak{M}_{N,n}(\mathbb{C}^3))|_{\epsilon=0} = (-1)^{Nn} \text{Eu}(T_{\tilde{\pi}} \mathfrak{M}_{N,n}(\mathbb{C}^3))|_{\epsilon=0},$$

and the partition function in this case is the generating function for higher rank Coulomb branch Donaldson–Thomas invariants of  $\mathbb{C}^3$  (D6–D0 states) given by the MacMahon function

$$\mathcal{Z}_{\mathbb{C}^3}^{\text{inst}}(\vec{\epsilon}, \vec{a}; \mathbf{q})|_{\epsilon=0} = M((-1)^N \mathbf{q})^N \quad \text{with} \quad M(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}, \quad (4.6)$$

independently of the equivariant parameters and the Higgs vevs. This is the six-dimensional version of the Vafa–Witten partition function (3.4) on  $\mathbb{C}^2$ . The simplicity of the Coulomb branch invariants in this case may be understood by rewriting them using the Joyce–Song formalism of generalized Donaldson–Thomas invariants  $\text{DT}(n)$ , as explained in [32]. In the present case they are defined through

$$\mathcal{Z}_{\mathbb{C}^3}^{\text{inst}}(\vec{\epsilon}, \vec{a}; \mathbf{q})|_{\epsilon=0} =: \exp \left( - \sum_{n=1}^{\infty} (-1)^{nN} n N \text{DT}(n) \mathbf{q}^n \right),$$

and they lead to the generalized Gopakumar–Vafa BPS invariants  $\text{BPS}(n)$  defined by

$$\text{DT}(n) =: \sum_{m|n} \frac{1}{m^2} \text{BPS}(n/m).$$

By taking the logarithm of the MacMahon function and resumming we can express it as

$$M(q) = \exp \left( \sum_{n=1}^{\infty} \frac{1}{(1-q^n)^2} \frac{q^n}{n} \right),$$

from which we find explicitly

$$\text{DT}(n) = \sum_{m|n} \frac{1}{m^2} \quad \text{and} \quad \text{BPS}(n) = 1.$$

We will now generalize the result (4.6) to arbitrary points  $\vec{\epsilon}$  in the  $\Omega$ -deformation; see §A.3 for the set up of the relevant combinatorial notation.

**Proposition 4.7** *The instanton partition function of  $\mathcal{N} = 2$  gauge theory on  $\mathbb{C}^3$  at a generic point  $\vec{\epsilon}$  of the  $\Omega$ -deformation is given by*

$$\mathcal{Z}_{\mathbb{C}^3}^{\text{inst}}(\vec{\epsilon}, \vec{a}; \mathbf{q}) = \sum_{\tilde{\pi}} ((-1)^N \mathbf{q})^{|\tilde{\pi}|} \prod_{l,l'=1}^N \frac{N_{l,l'}^{\tilde{\pi}}(\vec{\epsilon}, a_l - a_{l'})}{T_{l,l'}^{\tilde{\pi}}(\vec{\epsilon}, a_l - a_{l'})}, \quad (4.8)$$



where

$$\begin{aligned} N_{l,l'}^{\vec{\pi}}(\vec{\epsilon}, a) &= \prod_{(i,j,k) \in \pi_l} \left( (a + i \epsilon_1 + j \epsilon_2 + (k - (\pi_{l'})_{1,1}) \epsilon_3) \right. \\ &\quad \times \prod_{k'=1}^{(\pi_{l'})_{1,1}} (a + (i - (\pi_{l'}^{tt}(k'))_j) \epsilon_1 + ((\pi_l^t(k))_i - j + 1) \epsilon_2 + (k' - k + 1) \epsilon_3) \\ &\quad \left. \times (a + (i - (\pi_{l'}^{tt}(k'))_j - 1) \epsilon_1 + ((\pi_l^t(k))_i - j) \epsilon_2 + (k' - k - 1) \epsilon_3) \right) \end{aligned}$$

and

$$\begin{aligned} T_{l,l'}^{\vec{\pi}}(\vec{\epsilon}, a) &= \prod_{(i,j,k) \in \pi_l} \left( (a + (i - 1) \epsilon_1 + (j - 1) \epsilon_2 + (k - (\pi_{l'})_{1,1} - 1) \epsilon_3) \right. \\ &\quad \times \prod_{k'=1}^{(\pi_{l'})_{1,1}} (a + (i - (\pi_{l'}^{tt}(k'))_j - 1) \epsilon_1 + ((\pi_l^t(k))_i - j) \epsilon_2 + (k' - k) \epsilon_3) \\ &\quad \left. \times (a + (i - (\pi_{l'}^{tt}(k'))_j) \epsilon_1 + ((\pi_l^t(k))_i - j + 1) \epsilon_2 + (k' - k) \epsilon_3) \right). \end{aligned}$$

**Proof:** We write the decomposition of the  $\tilde{\mathbb{T}}$ -module  $V_{\vec{\pi}}$  in (4.4) as

$$V_{\vec{\pi}} = \sum_{l=1}^N e_l \sum_{k=1}^{(\pi_l)_{1,1}} R_{\pi_l^t(k)} t_3^{k-1} \quad \text{with} \quad R_{\pi_l^t(k)} = \sum_{(i,j) \in \pi_l^t(k)} t_1^{i-1} t_2^{j-1}.$$

Then the last term of the character (4.5) may be expressed as

$$\begin{aligned} &(t_1^{-1} - 1) (t_2^{-1} - 1) (t_3^{-1} - 1) V_{\vec{\pi}}^* \otimes V_{\vec{\pi}} \\ &= (t_3^{-1} - 1) \sum_{l,l'=1}^N e_l^{-1} e_{l'} \sum_{k=1}^{(\pi_l)_{1,1}} \sum_{k'=1}^{(\pi_{l'})_{1,1}} t_3^{k'-k} \left( (t_1^{-1} - 1) (t_2^{-1} - 1) R_{\pi_l^t(k)}^* \otimes R_{\pi_{l'}^t(k')} \right). \end{aligned}$$

Using the calculation of [82, Thm. 2.11] we have

$$(t_1^{-1} - 1) (t_2^{-1} - 1) R_{\pi_l^t(k)}^* \otimes R_{\pi_{l'}^t(k')} = R_{\pi_{l'}^t(k')} + (t_1 t_2)^{-1} R_{\pi_l^t(k)}^* - \mathcal{J}_{\pi_l^t(k), \pi_{l'}^t(k')}$$

where

$$\mathcal{J}_{\pi_l^t(k), \pi_{l'}^t(k')} = \sum_{(i,j) \in \pi_l^t(k)} t_1^{(\pi_{l'}^{tt}(k'))_j - i} t_2^{j - (\pi_l^t(k))_i - 1} + \sum_{(i,j) \in \pi_{l'}^t(k')} t_1^{i - (\pi_l^t(k))_j - 1} t_2^{(\pi_{l'}^t(k'))_i - j}.$$

Whence the full character (4.5) is given by

$$\begin{aligned} \text{Ch}(T_{\vec{\pi}}^{\text{vir}} \mathfrak{M}_{N,k}) &= V_{\vec{\pi}} \sum_{l=1}^N e_l^{-1} \left( 1 + (t_3^{-1} - 1) \sum_{k=1}^{(\pi_l)_{1,1}} t_3^{-k+1} \right) \\ &\quad - (t_1 t_2 t_3)^{-1} V_{\vec{\pi}}^* \sum_{l=1}^N e_l \left( 1 + (t_3 - 1) \sum_{k=1}^{(\pi_l)_{1,1}} t_3^{k-1} \right) \\ &\quad - (t_3^{-1} - 1) \sum_{l,l'=1}^N e_l^{-1} e_{l'} \sum_{k=1}^{(\pi_l)_{1,1}} \sum_{k'=1}^{(\pi_{l'})_{1,1}} t_3^{k'-k} \mathcal{J}_{\pi_l^t(k), \pi_{l'}^t(k')} \end{aligned}$$

$$\begin{aligned}
 &= V_{\vec{\pi}} \sum_{l=1}^N e_l^{-1} t_3^{-(\pi_l)_{1,1}} - (t_1 t_2 t_3)^{-1} V_{\vec{\pi}}^* \sum_{l=1}^N e_l t_3^{(\pi_l)_{1,1}} \\
 &\quad - (t_3^{-1} - 1) \sum_{l,l'=1}^N e_l^{-1} e_{l'} \sum_{k=1}^{(\pi_l)_{1,1}} \sum_{k'=1}^{(\pi_{l'})_{1,1}} t_3^{k'-k} \mathcal{T}_{\pi_l^t(k), \pi_{l'}^t(k')} \\
 &= \sum_{l,l'=1}^N e_l^{-1} e_{l'} \left( \sum_{(i,j,k) \in \pi_{l'}} t_1^{i-1} t_2^{j-1} t_3^{k-(\pi_l)_{1,1}-1} - \sum_{(i,j,k) \in \pi_l} t_1^{-i} t_2^{-j} t_3^{-k+(\pi_{l'})_{1,1}} \right. \\
 &\quad \left. - \sum_{k'=1}^{(\pi_{l'})_{1,1}} \sum_{(i,j,k) \in \pi_l} t_1^{(\pi_{l'}^t(k'))_{j-i}} t_2^{j-(\pi_l^t(k))_{i-1}} (t_3^{k'-k-1} - t_3^{k-k'}) \right. \\
 &\quad \left. - \sum_{k'=1}^{(\pi_l)_{1,1}} \sum_{(i,j,k) \in \pi_{l'}} t_1^{i-(\pi_l^t(k'))_{j-1}} t_2^{(\pi_{l'}^t(k))_{i-j}} (t_3^{k'-k-1} - t_3^{k'-k}) \right).
 \end{aligned}$$

Collecting and relabelling terms, this finally yields the expression for the ratio of fluctuation determinants in (4.8).  $\blacksquare$

It is easy to see that at the Calabi–Yau specialization one has

$$N_{l,l'}^{\vec{\pi}}(\vec{\epsilon}, a)|_{\epsilon=0} = T_{l,l'}^{\vec{\pi}}(\vec{\epsilon}, a)|_{\epsilon=0},$$

and hence that the partition function (4.8) reduces to (4.6) in this case. From this result it also follows that there are no non-trivial instanton contributions at the locus  $\epsilon_1 + \epsilon_2 = 0$  of the  $\Omega$ -deformation, as then

$$\mathcal{Z}_{\mathbb{C}^3}^{\text{inst}}(\sigma, -\sigma, \epsilon_3, \vec{a}; \mathbf{q}) = 1;$$

to see this, note that  $(i-j)\sigma + (k - (\pi_l)_{1,1})\epsilon_3$  vanishes at  $(i, j, k) = (1, 1, (\pi_l)_{1,1}) \in \pi_l$ , hence  $N_{l,l}^{\vec{\pi}}(\sigma, -\sigma, \epsilon_3, 0) = 0$  whenever  $\pi_l \neq \emptyset$ .

#### 4.2 $U(1)$ gauge theory

Remarkably, as in four dimensions one can sum the combinatorial series (4.8) explicitly in the rank one case. For  $N = 1$ , the moduli space  $\mathfrak{M}_{1,n}(\mathbb{C}^3) \cong \text{Hilb}^n(\mathbb{C}^3)$  is the Hilbert scheme of  $n$  points on  $\mathbb{C}^3$  and the computation of the equivariant instanton partition function is the content of [74, Thm. 1] which gives

$$\mathcal{Z}_{\mathbb{C}^3}^{\text{inst}}(\vec{\epsilon}; \mathbf{q}) = M(-\mathbf{q})^{-\chi_{\mathbb{T}^3}(\mathbb{C}^3)}, \quad (4.9)$$

where

$$\chi_{\mathbb{T}^3}(X) = \int_X \text{ch}_3^{\mathbb{T}^3}(TX) = \frac{(\epsilon_1 + \epsilon_2)(\epsilon_1 + \epsilon_3)(\epsilon_2 + \epsilon_3)}{\epsilon_1 \epsilon_2 \epsilon_3}$$

is the  $\mathbb{T}^3$ -equivariant Euler characteristic of  $X = \mathbb{C}^3$ , evaluated by the Bott residue formula. This formula is proven using geometric arguments from relative Donaldson–Thomas theory.

Our general result (4.8) then computes equivariant Coulomb branch invariants which may be regarded as a higher rank generalization of the degree zero equivariant Donaldson–Thomas invariants computed by (4.9). In fact, the conjectures of [84, 7] can be stated in our case as

**Conjecture 4.10** *The instanton partition function of  $N = 2$  gauge theory on  $\mathbb{C}^3$  at a generic point  $\vec{\epsilon}$  of the  $\Omega$ -deformation is given by*

$$\mathcal{Z}_{\mathbb{C}^3}^{\text{inst}}(\vec{\epsilon}, \vec{a}; \mathbf{q}) = M((-1)^N \mathbf{q})^{-N \chi_{\mathbb{T}^3}(\mathbb{C}^3)}.$$

#### 4.3 Reductions to four dimensions

In the remainder of this section we will present two applications of the “full”  $\Omega$ -deformation which relates the six-dimensional  $N = 2$  gauge theory to the four-dimensional  $N = 2$  gauge theories. Consider first the action of an additional complex torus  $\mathbb{T} = \mathbb{C}^\times$  whose generator  $t$  acts on the homogeneous coordinates of the projective space  $\mathbb{P}^3$  as

$$t \triangleright [w_0, w_1, w_2, w_3] := [w_0, w_1, w_2, t w_3] .$$

The fixed point set for this  $\mathbb{T}$ -action is just the linearly embedded projective plane  $(\mathbb{P}^3)^\mathbb{T} = \mathbb{P}^2$ . This  $\mathbb{T}$ -action commutes with the  $\mathbb{T}^3$ -action on  $\mathbb{P}^3$ , and, as a toric variety, in this reduction we restrict to the subtorus  $\mathbb{T}^2 \subset \mathbb{T}^3$  defined by  $t_3 = t_2^*$ , i.e.  $\epsilon_2 + \epsilon_3 = 0$ , which acts on the fixed  $\mathbb{P}^2$  in the standard way. In this limit the instanton partition function on  $\mathbb{C}^3$  is trivial, but we need to look more closely at its definition when defining a suitable reduction of it.

The linear algebraic form of the instanton deformation complex (4.3) is based on the ADHM-type parameterization of the moduli space as the GIT quotient [31]

$$\mathfrak{M}_{N,n}(\mathbb{C}^3) = \mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0) / GL(n, \mathbb{C}) \quad (4.11)$$

with

$$\mu_1 = [b_1, b_2] + i_1 i_2 , \quad \mu_2 = [b_1, b_3] + i_1 i_3 \quad \text{and} \quad \mu_3 = [b_2, b_3] . \quad (4.12)$$

Here  $(b_1, b_2, b_3) \in \text{End}_{\mathbb{C}}(V_{\vec{\pi}}) \otimes Q$ ,  $i_1 \in \text{Hom}_{\mathbb{C}}(W_{\vec{\pi}}, V_{\vec{\pi}})$  and  $(i_2, i_3, 0) \in \text{Hom}_{\mathbb{C}}(V_{\vec{\pi}}, W_{\vec{\pi}}) \otimes \bigwedge^2 Q$ , together with a stability condition analogous to that of the usual ADHM construction, i.e. the image of  $i_1$  generates  $V_{\vec{\pi}}$  under the action of  $b_1, b_2, b_3$ , and as before the group  $GL(n, \mathbb{C})$  acts by basis change automorphisms of the vector space  $V_{\vec{\pi}} \cong \mathbb{C}^n$ . The first map of (4.3) is a linearised  $GL(n, \mathbb{C})$  transformation, while the second map is the differential of the maps (4.12) that define the moduli space.

We shall now study the moduli of sheaves on  $\mathbb{P}^2$  which are pullbacks of framed torsion-free sheaves on  $\mathbb{P}^3$  by the inclusion  $\mathbb{P}^2 \subset \mathbb{P}^3$ . The  $\mathbb{T}$ -action on  $\mathbb{P}^3$  has a natural lift to the instanton moduli space  $\mathfrak{M}_{N,n}(\mathbb{C}^3)$  which is defined on the linear algebraic data as

$$t \triangleright (b_1, b_2, b_3, i_1, i_2, i_3) = (b_1, b_2, t b_3, i_1, i_2, t i_3) . \quad (4.13)$$

The subvariety of invariants consists of sextuples  $(b_1, b_2, 0, i_1, i_2, 0)$  which are identified with matrices  $(b_1, b_2) \in \text{End}_{\mathbb{C}}(V_{\vec{\pi}}) \otimes Q|_{\mathbb{T}^2}$ ,  $i_1 \in \text{Hom}_{\mathbb{C}}(W_{\vec{\pi}}, V_{\vec{\pi}})$  and  $i_2 \in \text{Hom}_{\mathbb{C}}(V_{\vec{\pi}}, W_{\vec{\pi}}) \otimes \bigwedge^2 Q|_{\mathbb{T}^2}$ . The  $\mathbb{T}$ -action (4.13) preserves the zero sets of the maps (4.12) and the action of the group  $GL(n, \mathbb{C})$ . The pullbacks of the maps  $\mu_2$  and  $\mu_3$  to the subvariety of invariants vanish, so their zero sets are trivial, while the pullback of  $\mu_1$  coincides with the ADHM moment map (2.10) after relabelling  $i := i_1$  and  $j := i_2$ . One can also easily check that the stability condition here pulls back to the one used in the ADHM construction of §2.4. Altogether we have shown that the moduli spaces of instantons in four dimensions can be described as smooth holomorphic submanifolds of the moduli spaces of generalized instantons in six dimensions,

$$\mathfrak{M}_{N,n}(\mathbb{C}^2) = \mathfrak{M}_{N,n}(\mathbb{C}^3)^\mathbb{T} .$$

We stress that this identification holds within the full  $U(N)$  gauge theory and not just on its Coulomb branch, insofar as the matrix equations (4.12) describe moduli of  $U(N)$  noncommutative instantons on  $\mathbb{C}^3$  [31]. Note how the full  $\Omega$ -deformation with  $\epsilon \neq 0$  is necessary here in order to reduce to a full  $\mathbb{T}^2$ -action in four dimensions.

One can extend these considerations to some of the other Calabi–Yau threefolds for which the generalised instanton moduli can be described in terms of linear algebraic data. This is the case, for example, for the  $A_{k-1}$ -surface singularity  $\mathbb{C}^2/\mathbb{Z}_k \times \mathbb{C}$  with the  $\mathbb{Z}_k$ -action given again by (2.7); its crepant resolution  $\text{Hilb}^{\mathbb{Z}_k}(\mathbb{C}^3)$  is a fibration of the minimal resolution  $X_k$  over the affine line  $\mathbb{C}$ . Its Coulomb branch partition functions at the Calabi–Yau specialisation  $\epsilon = 0$  are computed in [32, §7]. By the three-dimensional version of the McKay correspondence, the instanton moduli space is the moduli variety of representations of the associated McKay quiver, which in this instance coincides exactly with (2.24); the matrix equations are given by the  $\mathbb{Z}_k$ -equivariant decomposition of (4.12), analogously to the construction of §2.4. For  $\epsilon \neq 0$  one can now use the above construction to realise the Nakajima quiver varieties  $\mathfrak{M}_\xi(\vec{v}, \vec{w})$  as  $\mathbb{T}$ -invariant submanifolds of the moduli scheme of instantons on the (small radius) quotient stack  $[\mathbb{C}^2/\mathbb{Z}_k \times \mathbb{C}]$ , respectively the (large radius) Calabi–Yau resolution  $\text{Hilb}^{\mathbb{Z}_k}(\mathbb{C}^3)$ . It would be interesting to see if this six-dimensional perspective, and in particular the wall-crossing behaviour described in [32, §7.4], offers any insight into the chamber differences between the  $N = 2$  gauge theory partition functions discussed in §2 and hence to their associated AGT dualities. More generally, if  $X \subset Y$  is a surface embedded in a threefold  $Y$  as the fixed points of a  $\mathbb{T}$ -action, it would be interesting to develop the conditions required for the generalised instanton moduli on  $Y$  to correctly pullback to instanton moduli on  $X$ .

Our considerations are in harmony with the considerations of [16] which show that the partition function of the six-dimensional  $N = 2$  gauge theory on a local surface can be reduced to the partition function of Vafa–Witten theory on the base surface via equivariant localization with respect to the scaling action of  $\mathbb{T}$  on the fibers; it would be interesting to generalise this construction to the  $\Omega$ -deformed gauge theory. The construction also fits nicely with the embedding of  $U(N)$  Vafa–Witten theory on the ALE space  $X_k$  in Type IIA string theory as a D4–D6-brane intersection over a torus  $T^2$  [38], which also has a lift to M-theory on the compactification  $\text{TN}_N \times T^2 \times \mathbb{R}^5$ . The reductions described here are completely analogous to those of [89], who show that the BPS moduli space of the supersymmetric gauge theory on D4–D2–D0-branes on a noncompact toric divisor  $D$  of a toric Calabi–Yau threefold  $X$  can be embedded in the moduli space of BPS D6–D2–D0-brane states on  $X$ ; here the inclusion map is characterized combinatorially by a perfect matching in a brane tiling construction of the gauge theory on the D-branes (dimer model), and its image can be regarded as a  $\mathbb{T}$ -invariant subspace. It would be interesting to understand whether the  $\Omega$ -deformed  $N = 2$  gauge theories in four dimensions can be similarly understood via the twisted M-theory lifts discussed in §4.1, along the lines of the more general intersecting brane constructions considered in [38].

Another way to effectively achieve a dimensional reduction of the theory is via the analog of the Nekrasov–Shatashvili limit of four-dimensional  $N = 2$  supersymmetric gauge theory [88]; in our case this corresponds to the locus  $\epsilon_3 = 0$  of the  $\Omega$ -deformation. In this case the M-theory fibration  $\mathcal{M}_\rho$  from §4.1 contains an invariant line  $\mathbb{C}$  so we expect that the six-dimensional gauge theory in this limit reduces to some effective exactly solvable four-dimensional field theory, which may be thought of as arising through some sort of geometric engineering type construction via a level-rank duality analogously to the constructions of [38]. Unfortunately, the combinatorial expansion (4.8) is not useful for studying this limit, as it diverges at  $\epsilon_3 = 0$ . In analogy with the four-dimensional case, we can conjecture that the free energy  $\log \mathcal{Z}_{\mathbb{C}^3}^{\text{inst}}(\vec{\epsilon}, \vec{a}; \mathbf{q})$  has only simple monomial poles in the equivariant parameters, as in (4.9); they originate from the equivariant volume  $\int_X 1 = \frac{1}{\epsilon_1 \epsilon_2 \epsilon_3}$  of  $X = \mathbb{C}^3$ .

One way to extract the  $\epsilon_3 = 0$  limit analogously to the computations of [88] is to again exploit the parameterization (4.11) of the instanton moduli space and use the matrix model representation of the integral in (4.1) which was derived in [31, eq. (5.23)]; it is given by

$$\int_{[\mathfrak{M}_{N,n}(\mathbb{C}^3)]^{\text{vir}}} 1 = \frac{(-1)^n}{n!} \int_{\mathbb{R}^n} \prod_{i=1}^n \frac{d\phi_i}{2\pi i} \frac{P_N(\phi_i + \epsilon)}{P_N(\phi_i)} \prod_{1 \leq i < j \leq n} \mathcal{D}(\phi_i - \phi_j),$$

where

$$P_N(x) = \prod_{l=1}^N (x - a_l) ,$$

$$\mathcal{D}(x) = \frac{x(x + \epsilon_1 + \epsilon_2)(x + \epsilon_1 + \epsilon_3)(x + \epsilon_2 + \epsilon_3)}{(x + \epsilon)(x + \epsilon_1)(x + \epsilon_2)(x + \epsilon_3)} .$$

The integrand here has simple poles at the fixed points of the  $\tilde{\mathbb{T}}$ -action which are also parameterized by three-dimensional Young diagrams  $\vec{\pi}$  of size  $|\vec{\pi}| = n$  [31]. A suitable evaluation of this integral could be useful for extracting the  $\epsilon_3 = 0$  limit via an integral equation for the free energy analogously to that sketched in [88, §6.2.6]; note that  $\mathcal{D}(x) = 1$  in the limit  $\epsilon_3 = 0$ .

#### 4.4 Classical and perturbative partition functions

Let us now see how the effective four-dimensional theory comes about in the limit  $\epsilon_3 = 0$ . The full partition function of the six-dimensional cohomological gauge theory in the weak coupling regime receives classical, one-loop and instanton contributions

$$\mathcal{Z}_{\mathbb{C}^3}(\vec{\epsilon}, \vec{a}; \mathbf{q}) = \mathcal{Z}_{\mathbb{C}^3}^{\text{cl}}(\vec{\epsilon}, \vec{a}; \mathbf{q}) \mathcal{Z}_{\mathbb{C}^3}^{\text{pert}}(\vec{\epsilon}, \vec{a}) \mathcal{Z}_{\mathbb{C}^3}^{\text{inst}}(\vec{\epsilon}, \vec{a}; \mathbf{q}) .$$

The easiest way to extract the classical and perturbative partition functions is via a noncommutative deformation of the gauge theory. As concerns the instanton contributions, the noncommutative gauge theory exhibits the same properties as above and seems to offer no new information. Setting  $\epsilon_3 = 0$  implies that the noncommutative Higgs field  $\Phi$  vanishes on a one-particle subspace of the Fock space, hence the operator  $\exp(t\Phi)$  is not trace-class. Moreover, it is straightforward to modify the fluctuation integrals of [31, eq. (3.51)] at a generic point  $\vec{\epsilon}$  of the  $\Omega$ -deformation; the instanton integrals there then reproduce our previous expression for the instanton partition function (4.8).

The classical part of the partition function is given by [31, §3.5]

$$\mathcal{Z}_{\mathbb{C}^3}^{\text{cl}}(\vec{\epsilon}, \vec{a}; \mathbf{q}) = \prod_{l=1}^N \mathbf{q}^{-\frac{a_l^3}{6\epsilon_1\epsilon_2\epsilon_3}} .$$

The expression for the vacuum contribution in [31, eq. (3.51)] holds at all points in parameter space and gives the perturbative partition function

$$\mathcal{Z}_{\mathbb{C}^3}^{\text{pert}}(\vec{\epsilon}, \vec{a}) = \prod_{l,l'=1}^N \exp\left(-\int_0^\infty \frac{dt}{t} \frac{e^{t(a_l - a_{l'})}}{(1 - e^{t\epsilon_1})(1 - e^{t\epsilon_2})(1 - e^{t\epsilon_3})}\right) .$$

We can regularize the argument of the exponential function analogously to §2.2 using the function

$$\gamma_{\epsilon_1, \epsilon_2, \epsilon_3}(x; \Lambda) = \lim_{s \rightarrow 0} \frac{d}{ds} \frac{\Lambda^s}{\Gamma(s)} \int_0^\infty dt t^{s-1} \frac{e^{-tx}}{(1 - e^{t\epsilon_1})(1 - e^{t\epsilon_2})(1 - e^{t\epsilon_3})} \quad (4.14)$$

with  $x = a_{l'} - a_l$ . We expand

$$\frac{1}{(1 - e^{t\epsilon_1})(1 - e^{t\epsilon_2})(1 - e^{t\epsilon_3})} = \sum_{n=0}^\infty \frac{c_n}{n!} t^{n-3} , \quad (4.15)$$

where the first few terms are given by

$$\begin{aligned} c_0 &= -\frac{1}{\epsilon_1 \epsilon_2 \epsilon_3}, \\ c_1 &= \frac{\epsilon_1 + \epsilon_2 + \epsilon_3}{2 \epsilon_1 \epsilon_2 \epsilon_3}, \\ c_2 &= -\frac{\epsilon_1^2 + 3(\epsilon_2 + \epsilon_3)\epsilon_1 + \epsilon_2^2 + \epsilon_3^2 + 3\epsilon_2 \epsilon_3}{6 \epsilon_1 \epsilon_2 \epsilon_3}, \\ c_3 &= \frac{(\epsilon_1 + \epsilon_2 + \epsilon_3)(\epsilon_2 \epsilon_3 + \epsilon_1(\epsilon_2 + \epsilon_3))}{4 \epsilon_1 \epsilon_2 \epsilon_3}. \end{aligned}$$

It is easy to check that the coefficients  $c_n$  all have denominator  $\epsilon_1 \epsilon_2 \epsilon_3$ . We can rescale  $t \rightarrow \frac{t}{x}$  to generate gamma-functions and write

$$\gamma_{\epsilon_1, \epsilon_2, \epsilon_3}(x; \Lambda) = \lim_{s \rightarrow 0} \frac{d}{ds} \frac{\Lambda^s}{x^s} \sum_{n=0}^{\infty} \frac{c_n}{n!} x^{3-n} \frac{\Gamma(n+s-3)}{\Gamma(s)},$$

and taking the limit explicitly finally gives

$$\begin{aligned} \gamma_{\epsilon_1, \epsilon_2, \epsilon_3}(x; \Lambda) &= c_0 x^3 \left( -\frac{11}{36} + \frac{1}{6} \log \frac{x}{\Lambda} \right) + c_1 x^2 \left( \frac{3}{4} - \frac{1}{2} \log \frac{x}{\Lambda} \right) + c_2 \frac{x}{2} \left( -1 + \log \frac{x}{\Lambda} \right) \\ &\quad - \frac{c_3}{6} \log \frac{x}{\Lambda} + \sum_{n=4}^{\infty} \frac{1}{n(n-1)(n-2)(n-3)} c_n x^{3-n}. \end{aligned}$$

Finally the perturbative part of the partition function is given by

$$\mathcal{Z}_{\mathbb{C}^3}^{\text{pert}}(\vec{c}, \vec{a}) = \exp \left( - \sum_{l, l'=1}^N \gamma_{\epsilon_1, \epsilon_2, \epsilon_3}(a_{l'} - a_l; \Lambda) \right),$$

which is analogous to the one-loop partition function (2.5) in four dimensions. However, here the appearance of logarithmic functions in this construction is somewhat puzzling; in the four-dimensional case they arise because the gauge theory is asymptotically free, however here the gauge theory is topological (hence automatically infrared free). Similarly the meaning of the cutoff  $\Lambda$  is unclear, since it appears as a Yang–Mills scale while it should be an ultraviolet cutoff.

The function  $\gamma_{\epsilon_1, \epsilon_2, \epsilon_3}(x; \Lambda)$  has interesting properties in the limit  $\epsilon_3 \rightarrow 0$ . Let us define

$$\Pi_{\epsilon_1, \epsilon_2}(x; \Lambda) := \lim_{\epsilon_3 \rightarrow 0} \epsilon_3 \gamma_{\epsilon_1, \epsilon_2, \epsilon_3}(x; \Lambda).$$

The analogous quantity  $\gamma_{\epsilon_1, \epsilon_2}(x; \Lambda)$  in the four-dimensional case replaces (4.15) with

$$\frac{1}{(1 - e^{t\epsilon_1})(1 - e^{t\epsilon_2})} = \sum_{n=0}^{\infty} \frac{d_n}{n!} t^{n-2}.$$

Multiplying (4.15) by  $\epsilon_3$  and taking the limit  $\epsilon_3 \rightarrow 0$ , by using l'Hôpital's rule

$$\lim_{\epsilon_3 \rightarrow 0} \frac{\epsilon_3}{1 - e^{t\epsilon_3}} = -\frac{1}{t}$$

we derive the relationship

$$\lim_{\epsilon_3 \rightarrow 0} \epsilon_3 c_n(\epsilon_1, \epsilon_2, \epsilon_3) = -d_n(\epsilon_1, \epsilon_2). \quad (4.16)$$

This also shows that

$$\frac{d}{dx} \Pi_{\epsilon_1, \epsilon_2}(x; \Lambda) = \gamma_{\epsilon_1, \epsilon_2}(x; \Lambda) ,$$

where according to [81, eq. (E.3)] one has

$$\begin{aligned} \gamma_{\epsilon_1, \epsilon_2}(x; \Lambda) &= \frac{1}{\epsilon_1 \epsilon_2} \left( -\frac{1}{2} x^2 \log \frac{x}{\Lambda} + \frac{3}{4} x^2 \right) + \frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \left( -x \log \frac{x}{\Lambda} + x \right) \\ &\quad - \frac{\epsilon_1^2 + \epsilon_2^2 + 3 \epsilon_1 \epsilon_2}{12 \epsilon_1 \epsilon_2} \log \frac{x}{\Lambda} + \sum_{n=3}^{\infty} \frac{d_n x^{2-n}}{n(n-1)(n-2)} . \end{aligned}$$

Mimicking the approach of [88], we define the quantity

$$\mathcal{W}_{\mathbb{C}^3}(\epsilon_1, \epsilon_2, \vec{a}; \mathbf{q}) := \lim_{\epsilon_3 \rightarrow 0} \epsilon_3 \log \mathcal{Z}_{\mathbb{C}^3}(\vec{\epsilon}, \vec{a}; \mathbf{q}) ,$$

which receives a sum of classical, one-loop and instanton contributions. We have

$$\mathcal{W}_{\mathbb{C}^3}^{\text{cl}}(\epsilon_1, \epsilon_2, \vec{a}; \mathbf{q}) = - \sum_{l=1}^N \frac{a_l^3}{6 \epsilon_1 \epsilon_2} \log \mathbf{q}$$

and

$$\mathcal{W}_{\mathbb{C}^3}^{\text{pert}}(\epsilon_1, \epsilon_2, \vec{a}) = - \lim_{\epsilon_3 \rightarrow 0} \sum_{l, l'=1}^N \epsilon_3 \gamma_{\epsilon_1, \epsilon_2, \epsilon_3}(a_{l'} - a_l; \Lambda) = - \sum_{l, l'=1}^N \Pi_{\epsilon_1, \epsilon_2}(a_{l'} - a_l; \Lambda) .$$

Thus dropping signs the full semiclassical superpotential is then

$$\mathcal{W}_{\mathbb{C}^3}^{\text{sc}}(\epsilon_1, \epsilon_2, \vec{a}; \mathbf{q}) = \frac{1}{6 \epsilon_1 \epsilon_2} \log \mathbf{q} \sum_{l=1}^N a_l^3 + \sum_{l, l'=1}^N \Pi_{\epsilon_1, \epsilon_2}(a_{l'} - a_l; \Lambda) .$$

The relevant Bethe-type equations are obtained by minimizing this superpotential with respect to each of the variables  $a_l$  for  $l = 1, \dots, N$ , which gives

$$\frac{1}{2 \epsilon_1 \epsilon_2} \log \mathbf{q} a_l^2 + \sum_{l' \neq l} \gamma_{\epsilon_1, \epsilon_2}(a_l - a_{l'}; \Lambda) - \sum_{l' \neq l} \gamma_{\epsilon_1, \epsilon_2}(a_{l'} - a_l; \Lambda) = 0 ,$$

and by exponentiation our final equations are

$$\mathbf{q}^{\frac{a_l^2}{2 \epsilon_1 \epsilon_2}} = \prod_{l' \neq l} \frac{e^{\gamma_{\epsilon_1, \epsilon_2}(a_{l'} - a_l; \Lambda)}}{e^{\gamma_{\epsilon_1, \epsilon_2}(a_l - a_{l'}; \Lambda)}}$$

for  $l = 1, \dots, N$ . These Bethe-type equations yield an alternative regularized version of the formal expression (2.6) in terms of the contributions to the classical four-dimensional partition function (2.4).

In contrast to the context of four-dimensional gauge theory, the minimization of the superpotential  $\mathcal{W}_{\mathbb{C}^3}(\epsilon_1, \epsilon_2, \vec{a}; \mathbf{q})$  here does not appear to have a clear physical meaning; for this, one should probably look directly at the dimensional reduction of the physical untwisted gauge theory in six dimensions, which may have a more transparent meaning through its twisted M-theory lift discussed in §4.1. Now one can add instanton corrections and define similarly a nonperturbative superpotential, which by *assuming* Conj. 4.10 is given explicitly by

$$\mathcal{W}_{\mathbb{C}^3}^{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; \mathbf{q}) = N(\epsilon_1 + \epsilon_2) \sum_{n=1}^{\infty} \frac{1}{(1 - \mathbf{q}^n)^2} \frac{\mathbf{q}^n}{n} .$$



## Acknowledgments

We thank Alastair Craw, Elizabeth Gasparim, Iain Gordon, Melissa Liu and Paul Smith for helpful discussions, correspondence, and technical assistance on some of the issues discussed in this paper, and Ugo Bruzzo, Michele Cirafici, Lucio Cirio, Zoltán Kökenyesi, Giovanni Landi, Mattia Pedrini, Francesco Sala and Annamaria Sinkovics for collaboration on various parts of the material presented here. We also thank Ugo Bruzzo, Dimitri Markushevich, Vladimir Rubtsov, Francesco Sala and Sergey Shadrin for the invitation to contribute this article to the special issue, which is based on themes from the workshop “Instanton Counting: Moduli Spaces, Representation Theory and Integrable Systems” which was held at the Lorentz Center in June 2014. This work was supported in part by the Consolidated Grant ST/L000334/1 from the UK Science and Technology Facilities Council (STFC) and by the Action MP1405 QSPACE from the European Cooperation in Science and Technology (COST).

## A Combinatorics

### A.1 Young diagrams

A *partition* of a positive integer  $n$  is a nonincreasing sequence of positive numbers  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0)$  such that  $|\lambda| := \sum_{i=1}^{\ell} \lambda_i = n$ . We call  $\ell = \ell(\lambda)$  the *length* of the partition  $\lambda$ . A partition  $\lambda$  of  $n$  may also be described as a list  $\lambda = (1^{m_1} 2^{m_2} \dots)$ , where  $m_i = \#\{l \in \mathbb{Z}_{>0} \mid \lambda_l = i\}$ . Then  $\sum_i i m_i = n$  and  $\sum_i m_i = \ell$ . We also write  $\|\lambda\|^2 := \sum_{i=1}^{\ell} \lambda_i^2$ .

One can associate to a partition  $\lambda$  a *Young diagram*, which is the set  $Y_\lambda = \{(a, b) \in \mathbb{Z}_{>0}^2 \mid 1 \leq a \leq \ell, 1 \leq b \leq \lambda_a\}$ . Thus  $\lambda_a$  is the length of the  $a$ -th column of  $Y_\lambda$ ; we write  $|Y_\lambda| = |\lambda|$  for the *weight* of the Young diagram  $Y_\lambda$ . We identify a partition  $\lambda$  with its Young diagram  $Y_\lambda$ . For a partition  $\lambda$ , the *transpose partition*  $\lambda^t$  is the partition with Young diagram  $Y_{\lambda^t} := \{(j, i) \in \mathbb{Z}_{>0}^2 \mid (i, j) \in \lambda\}$ . On the set  $\Pi$  of all Young diagrams there is a natural partial ordering called *dominance ordering*: For two partitions  $\mu$  and  $\lambda$ , we write  $\mu \leq \lambda$  if  $|\mu| = |\lambda|$  and  $\mu_1 + \dots + \mu_i \leq \lambda_1 + \dots + \lambda_i$  for all  $i \geq 1$ . We write  $\mu < \lambda$  if  $\mu \leq \lambda$  and  $\mu \neq \lambda$ .

We call the elements of a Young diagram  $Y$  the *nodes* of  $Y$ . For a node  $s = (a, b) \in Y$ , we call the *arm-length* of  $s$  the quantity  $A(s) := A_Y(s) = \lambda_a - b$  and the *leg-length* of  $s$  the quantity  $L(s) := L_Y(s) = \lambda_b^t - a$ . The *hook-length* of  $s$  is  $h(s) := h_Y(s) = A_Y(s) + L_Y(s) + 1$ . The *content* of a node  $s = (a, b)$  is the number  $a - b$ . The *arm-colength* and *leg-colength* are respectively given by  $A^t(s) := A_Y^t(s) = b - 1$  and  $L^t(s) := L_Y^t(s) = a - 1$ . The *hook* of  $s$  is the set  $H_s := H_s(Y) = \{(c, d) \in Y \mid c = a, d \geq b\} \cup \{(c, d) \in Y \mid c > a, d = b\}$ ; then  $h(s) = \#H_s$  and we say that  $H_s$  is an  $r$ -hook if  $h(s) = r$ . For  $\nu \in \mathbb{Z}$ , we define  $N_\nu(Y)$  to be the number of nodes of  $Y$  along the line  $b = a - \nu$ .

Fix an integer  $k \geq 2$  and let  $i \in \{0, 1, \dots, k-1\}$ . A node of a Young diagram  $Y$  is called an  $i$ -node if its content equals  $i$  modulo  $k$ ; this gives a  $k$ -colouring of  $Y$ . We define

$$\begin{aligned} n_j(Y) &= \#\{(a, b) \in Y \mid a - b = j\}, \\ \nu_i(Y) &= \#\{(a, b) \in Y \mid a - b \equiv i \pmod{k}\}, \quad \vec{\nu}(Y) = (\nu_0(Y), \nu_1(Y), \dots, \nu_{k-1}(Y)) \in \mathbb{Z}_{\geq 0}^k. \end{aligned}$$

### A.2 Maya diagrams

A *Maya diagram* is an increasing sequence of half-integers  $\mathbf{m} := (h_j)_{j \geq 1}$  such that  $h_{j+1} = h_j + 1$  for sufficiently large  $j$ . Let  $\mathcal{M}$  denote the set of all Maya diagrams. Any  $\mathbf{m} \in \mathcal{M}$  can be identified

with a map  $\mathbf{m} : \mathbb{Z} + \frac{1}{2} \rightarrow \{\pm 1\}$  such that

$$\mathbf{m}(h) = \begin{cases} 1 & \text{for } h \gg 0, \\ -1 & \text{for } h \ll 0. \end{cases}$$

We define the *charge* of  $\mathbf{m}$  by  $c(\mathbf{m}) := \#\{h < 0 \mid \mathbf{m}(h) = 1\} - \#\{h > 0 \mid \mathbf{m}(h) = -1\}$ . For  $z \in \mathbb{Z}$ , we define  $\mathbf{m}^{(z)}$  by  $\mathbf{m}^{(z)}(h) := \mathbf{m}(h + z)$ ; then  $c(\mathbf{m}^{(z)}) = c(\mathbf{m}) - z$ .

We shall now explain how Maya diagrams can be identified with Young diagrams. Let  $Y$  be a Young diagram, and note that for any half-integer  $h$  one has

$$n_{h-\frac{1}{2}}(Y) - n_{h+\frac{1}{2}}(Y) = \begin{cases} -1 & \text{or } 0 & \text{for } h < 0, \\ 0 & \text{or } 1 & \text{for } h > 0. \end{cases}$$

We then define

$$\mathbf{m}_Y(h) = \begin{cases} -1 & \text{for } h < 0 \text{ and } n_{h-\frac{1}{2}}(Y) - n_{h+\frac{1}{2}}(Y) = 0, \\ 1 & \text{for } h < 0 \text{ and } n_{h-\frac{1}{2}}(Y) - n_{h+\frac{1}{2}}(Y) = -1, \\ -1 & \text{for } h > 0 \text{ and } n_{h-\frac{1}{2}}(Y) - n_{h+\frac{1}{2}}(Y) = 1, \\ 1 & \text{for } h > 0 \text{ and } n_{h-\frac{1}{2}}(Y) - n_{h+\frac{1}{2}}(Y) = 0. \end{cases}$$

Then  $c(\mathbf{m}_Y) = 0$ . Conversely, given a Maya diagram  $\mathbf{m}$  of zero charge, there exists a unique Young diagram  $Y$  such that  $\mathbf{m}_Y = \mathbf{m}$ . Consequently there is a bijection

$$F : \mathbb{Z} \times \Pi \longrightarrow \mathcal{M}, \quad (z, Y) \longmapsto \mathbf{m}_Y^{(-z)}.$$

Define  $q(\mathbf{m}) \in \Pi$  by  $F^{-1}(\mathbf{m}) = (c(\mathbf{m}), q(\mathbf{m}))$ .

Let  $\tilde{I} = \{\frac{1}{2}, \frac{3}{2}, \dots, k - \frac{1}{2}\}$ . For  $h \in \tilde{I}$  and  $\mathbf{m} \in \mathcal{M}$ , we define a Maya diagram  $\mathbf{m}_h$  by  $\mathbf{m}_h(l) = \mathbf{m}(k(l - \frac{1}{2}) + h)$ . Then  $\mathbf{m}$  can be recovered from  $\{\mathbf{m}_h\}_{h \in \tilde{I}}$  and  $c(\mathbf{m}) = \sum_{h \in \tilde{I}} c(\mathbf{m}_h)$ . For a Young diagram  $Y$ , we set  $c_h(Y) = c(\mathbf{m}_{Y,h})$  and  $q_h(Y) = q(\mathbf{m}_{Y,h})$ . By [77, Lem. 2.5.2], one has explicitly  $c_h(Y) = \nu_{h-\frac{1}{2}}(Y) - \nu_{h+\frac{1}{2}}(Y)$ . We define the  $k$ -core of  $Y$  by

$$\vec{c}(Y) = (c_h(Y))_{h \in \tilde{I}} \in (\mathbb{Z}^{\tilde{I}})_0 := \left\{ (c_{\frac{1}{2}}, \dots, c_{k-\frac{1}{2}}) \in \mathbb{Z}^{\tilde{I}} \mid \sum_{h \in \tilde{I}} c_h = 0 \right\}.$$

The set  $(\mathbb{Z}^{\tilde{I}})_0$  of  $k$ -cores is in bijection with the set of Young diagrams which do not contain any  $k$ -hooks. We also define the  $k$ -quotient of  $Y$  by  $\vec{q}(Y) = (q_h(Y))_{h \in \tilde{I}} \in \Pi^{\tilde{I}}$ . There is a bijection

$$\text{CQ} : \Pi \longrightarrow (\mathbb{Z}^{\tilde{I}})_0 \times \Pi^{\tilde{I}}, \quad Y \longmapsto (\vec{c}(Y), \vec{q}(Y)),$$

which is obtained by removing all  $k$ -hooks from a Young diagram  $Y$ ; the set of removed  $k$ -hooks uniquely constitutes a  $k$ -tuple of partitions  $(\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(k)})$  which corresponds to the  $k$ -quotient  $\vec{q}(Y)$ . In particular, for any element  $((c_h)_{h \in \tilde{I}}, (Y_h)_{h \in \tilde{I}}) \in (\mathbb{Z}^{\tilde{I}})_0 \times \Pi^{\tilde{I}}$  we have

$$\text{CQ}^{-1}((c_h)_{h \in \tilde{I}}, (Y_h)_{h \in \tilde{I}}) = q(\mathbf{m}),$$

where  $\mathbf{m}$  is the Maya diagram recovered from the Maya diagrams  $\mathbf{m}_{Y_h}$  associated to the Young diagrams  $Y_h$  for  $h \in \tilde{I}$ . The weight of  $Y \in \Pi$  is given by

$$|Y| = |\underline{Y}| + k |\vec{q}(Y)|$$

where  $\underline{Y} = \text{CQ}^{-1}(\vec{c}(Y), \vec{0})$ . If  $\vec{q}(Y) = \vec{0}$  then

$$|Y| = \sum_{h \in \tilde{I}} \sum_{l \in \mathbb{Z}} N_{kl+h-\frac{1}{2}}(Y) = \sum_{h \in \tilde{I}} \left( \frac{1}{2} k c_h(Y)^2 + (h - \frac{1}{2}) c_h(Y) \right).$$

On the other hand, if  $\vec{c}(Y) = \vec{0}$  then  $\underline{Y} = \emptyset$  and the  $k$ -quotient  $(\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(k)})$  can be read off from the relation

$$(\lambda_i - i)_{i \geq 1} = \bigcup_{r=0}^{k-1} \left( k \left( \lambda_{i_r}^{(r)} - i_r \right) + r \right)_{i_r \geq 1},$$

where the left-hand side is the corresponding *blended* partition while the right-hand side contains the corresponding *coloured* Young diagrams.

### A.3 Three-dimensional Young diagrams

A *three-dimensional Young diagram* can be regarded as a stack of ordinary (two-dimensional) Young diagrams, with boxes piled in the positive octant  $(x, y, z) \in \mathbb{Z}_{\geq 0}^3$ . We can construct an array of non-negative decreasing integers  $\pi = (\pi_{i,j})_{i,j \geq 1}$ ,  $\pi_{i,j} \geq \pi_{i+1,j}$ ,  $\pi_{i,j} \geq \pi_{i,j+1}$ , called a *plane partition*, as the height function of the stack of cubes defined on the  $(x, y)$  plane, i.e. the three-dimensional Young diagram (that we also denote by  $\pi$  for brevity) is the set  $\pi = \{(i, j, k) \in \mathbb{Z}_{>0}^3 \mid 1 \leq k \leq \pi_{i,j}\}$ . Its projection to the  $(x, y)$  plane defines an ordinary partition  $\pi^z$ . More generally, for fixed  $a, b, p, r \geq 0$  the sequence  $\lambda = (\lambda_i) := (\pi_{a+pi, b+ri})$  defines a partition. The *size* of  $\pi$  is

$$|\pi| := \sum_{(i,j) \in \pi^z} \pi_{i,j}.$$

We also obtain different arrays of non-negative integers  $\pi^t = (\pi_{j,k}^t)_{j,k \geq 1}$  and  $\pi^{tt} = (\pi_{i,k}^{tt})_{i,k \geq 1}$  by considering the height functions of the stack relative to the  $(y, z)$  and  $(x, z)$  planes, respectively; these transformations are the analogs of the transpose operation for two-dimensional Young diagrams. Again their respective projections to the  $(y, z)$  and  $(x, z)$  coordinate planes defines ordinary partitions  $\pi^x$  and  $\pi^y$ . For each fixed integer  $k \geq 1$ , the sequences  $\pi_{(k)}^t := (\pi_{j,k}^t)_{j \geq 1}$  and  $\pi_{(k)}^{tt} := (\pi_{i,k}^{tt})_{i \geq 1}$  are two-dimensional Young diagrams obtained by cutting the three-dimensional Young diagram  $\pi$  by the plane  $z = k$  perpendicular to the  $z$ -axis.

## B Symmetric functions

### B.1 Monomial symmetric functions

The algebra of symmetric polynomials in  $N$  variables is the subspace  $\Lambda_N := \mathbb{C}[x_1, \dots, x_N]^{S_N}$  of polynomials invariant under the action of the symmetric group of permutations  $S_N$ . It is a graded ring  $\Lambda_N = \bigoplus_{n \geq 0} \Lambda_N^n$ , where  $\Lambda_N^n$  is the ring of homogeneous symmetric polynomials in  $N$  variables of degree  $n$ . There are graded inclusion morphisms  $\Lambda_{N+1} \hookrightarrow \Lambda_N$  by setting  $x_{N+1} = 0$ , and the corresponding inverse inductive limit is the algebra of symmetric functions in infinitely many variables  $\Lambda := \bigoplus_{n \geq 0} \Lambda^n$ .

For a partition  $\mu = (\mu_1, \dots, \mu_t)$  with  $t \leq N$ , we define the polynomial

$$m_\mu(x_1, \dots, x_N) = \sum_{\sigma \in S_N} x_1^{\mu_{\sigma_1}} \cdots x_N^{\mu_{\sigma_N}},$$

where we set  $\mu_j = 0$  for  $j = t+1, \dots, N$ . The polynomial  $m_\mu$  is symmetric, and the set of all  $m_\mu$  for all partitions  $\mu$  with  $|\mu| \leq N$  is a basis of  $\Lambda_N$ . Then the collection of  $m_\mu$ , for all partitions  $\mu$  with  $|\mu| \leq N$  and  $\sum_i \mu_i = n$ , is a basis of  $\Lambda_N^n$ . Using the definition of inverse limit we can define the *monomial symmetric functions*  $m_\mu$ ; by varying over partitions  $\mu$  of  $n$ , these functions form a basis for  $\Lambda^n$ .

The  $n$ -th *power sum symmetric function* is  $p_n := m_{(n)} = \sum_i x_i^n$ . The collection of symmetric functions  $p_\mu := p_{\mu_1} \cdots p_{\mu_t}$ , for all partitions  $\mu = (\mu_1, \dots, \mu_t)$ , is another basis of  $\Lambda$ .

## B.2 Macdonald functions

Fix parameters  $q, t \in \mathbb{C}$  with  $|q| < 1$ . For  $a \in \mathbb{C}$ , we use throughout the standard hypergeometric notation for the infinite  $q$ -shifted factorial

$$(a; q)_\infty := \prod_{n=0}^{\infty} (1 - a q^n) .$$

Define an inner product on the vector space  $\Lambda \otimes \mathbb{Q}(q, t)$  with respect to which the basis of power sum symmetric functions  $p_\lambda(x)$  are orthogonal with the normalization

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda,\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}} ,$$

where  $\delta_{\lambda,\mu} := \prod_i \delta_{\lambda_i, \mu_i}$  and

$$z_\lambda := \prod_{j \geq 1} j^{m_j} m_j! .$$

This is called the *Macdonald inner product*.

The monic form of the *Macdonald functions*  $M_\lambda(x; q, t) \in \Lambda \otimes \mathbb{Q}(q, t)$  for  $x = (x_1, x_2, \dots)$  are uniquely defined by the following two conditions [70, Ch. VI]:

- (i) Triangular expansion in the basis  $m_\mu(x)$  of monomial symmetric functions:

$$M_\lambda(x; q, t) = m_\lambda(x) + \sum_{\mu < \lambda} v_{\lambda,\mu}(q, t) m_\mu(x) \quad \text{with} \quad v_{\lambda,\mu}(q, t) \in \mathbb{C} . \quad (\text{B.1})$$

- (ii) Orthogonality:

$$\langle M_\lambda, M_\mu \rangle_{q,t} = \delta_{\lambda,\mu} \prod_{s \in Y_\lambda} \frac{1 - q^{A(s)+1} t^{L(s)}}{1 - q^{A(s)} t^{L(s)+1}} . \quad (\text{B.2})$$

For  $t = 1$  these functions coincide with the monomial symmetric functions,  $M_\lambda(x; q, 1) = m_\lambda(x)$ .

By their definition the Macdonald functions are homogeneous:

$$M_\lambda(\zeta x; q, t) = \zeta^{|\lambda|} M_\lambda(x; q, t) \quad \text{for} \quad \zeta \in \mathbb{C} , \quad (\text{B.3})$$

and they satisfy the *Macdonald specialization identity*

$$M_\lambda(t^\rho; q, t) = q^{\frac{1}{4} \|\lambda\|^2} t^{-\frac{1}{4} \|\lambda^t\|^2} \left( \frac{t}{q} \right)^{\frac{|\lambda|}{4}} \prod_{s \in Y_\lambda} \left( q^{\frac{A(s)}{2}} t^{\frac{L(s)+1}{2}} - q^{-\frac{A(s)}{2}} t^{-\frac{L(s)+1}{2}} \right)^{-1} , \quad (\text{B.4})$$

where  $t^\rho := (t^{\rho_1}, t^{\rho_2}, \dots)$  with  $\rho_i = -i + \frac{1}{2}$ . If we take  $x = (t^{-1}, \dots, t^{-N}, 0, 0, \dots)$  for  $N \in \mathbb{Z}_{>0}$  then we can also write the special value

$$M_\lambda(t^{-1}, \dots, t^{-N}; q, t) = \prod_{s \in Y_\lambda} \frac{t^{L^t(s)-N} - q^{A^t(s)}}{1 - q^{A(s)} t^{L(s)+1}} \quad (\text{B.5})$$

for the Macdonald polynomials  $M_\lambda(x_1, \dots, x_N; q, t)$  in  $\Lambda_N \otimes \mathbb{Q}(q, t)$ .

Macdonald functions also satisfy the *generalized Cauchy–Binet formula*

$$\sum_{\lambda} \frac{1}{\langle M_{\lambda}, M_{\lambda} \rangle_{q,t}} M_{\lambda}(x; q, t) M_{\lambda}(y; q, t) = \prod_{i,j \geq 1} \frac{(t x_i y_j; q)_{\infty}}{(x_i y_j; q)_{\infty}}, \quad (\text{B.6})$$

where the sum runs over all partitions  $\lambda$ . By taking the logarithm of this expression and resummung, we can rewrite it in the form of a *generalized Cauchy–Stanley identity*

$$\sum_{\lambda} \frac{1}{\langle M_{\lambda}, M_{\lambda} \rangle_{q,t}} M_{\lambda}(x; q, t) M_{\lambda}(y; q, t) = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{1-t^n}{1-q^n} p_n(x) p_n(y) \right). \quad (\text{B.7})$$

### B.3 Uglov functions

Let us consider now the limit

$$q = \omega p, \quad t = \omega p^{\beta} \quad \text{with } p \rightarrow 1,$$

where  $\omega := e^{2\pi i/k}$  is a primitive  $k$ -th root of unity with  $k \in \mathbb{Z}_{>0}$  and  $\beta \in \mathbb{C}$ . The resulting symmetric functions are called the *rank  $k$  Uglov functions* or  *$\mathfrak{gl}_k$ -Jack functions* and are denoted by

$$U_{\lambda}(x; \beta, k) := \lim_{p \rightarrow 1} M_{\lambda}(x; \omega p, \omega p^{\beta}).$$

They were first introduced in [105].

The Uglov functions of rank  $k = 1$  are just the usual *monic Jack functions*

$$J_{\lambda}(x; \beta) = U_{\lambda}(x; \beta, 1)$$

in  $\Lambda \otimes \mathbb{Q}(\beta)$ . Taking the limit  $p \rightarrow 1$  in the inner product  $\langle -, - \rangle_{p,p^{\beta}}$  yields an inner product  $\langle -, - \rangle_{\beta}$  on  $\Lambda \otimes \mathbb{Q}(\beta)$  with

$$\langle p_{\lambda}, p_{\mu} \rangle_{\beta} = \delta_{\lambda,\mu} z_{\lambda} \beta^{-\ell(\lambda)}.$$

The orthogonality relation for the Jack functions thus reads

$$\langle J_{\lambda}, J_{\mu} \rangle_{\beta} = \delta_{\lambda,\mu} \prod_{s \in Y_{\lambda}} \frac{A(s) + 1 + \beta L(s)}{A(s) + \beta (L(s) + 1)}.$$

The homogeneity property (B.3) in this case becomes

$$J_{\lambda}(\zeta y; \beta) = \zeta^{|\lambda|} J_{\lambda}(y; \beta) \quad \text{for } \zeta \in \mathbb{C}.$$

Moreover, the Macdonald specialisation identities (B.4) and (B.5) in this limit become

$$\begin{aligned} J_{\lambda}(1, 1, \dots; \beta) &= \prod_{s \in Y_{\lambda}} \frac{1}{A(s) + \beta (L(s) + 1)}, \\ J_{\lambda}(\underbrace{1, \dots, 1}_{N \text{ times}}; \beta) &= \prod_{s \in Y_{\lambda}} \frac{A^t(s) - \beta (L^t(s) - N)}{A(s) + \beta (L(s) + 1)}, \end{aligned}$$

while the generalised Cauchy–Stanley identities read (cf. [99, Prop. 2.1])

$$\sum_{\lambda} \frac{1}{\langle J_{\lambda}, J_{\lambda} \rangle_{\beta}} J_{\lambda}(x; \beta) J_{\lambda}(y; \beta) = \prod_{i,j \geq 1} \frac{1}{(1 - x_i y_j)^{\beta}} = \exp \left( \beta \sum_{n=1}^{\infty} \frac{1}{n} p_n(x) p_n(y) \right).$$

For  $\beta = 1$ , the Jack functions reduce to the *Schur functions* (cf. [70, Ch. I, §3])

$$s_\lambda(x) = J_\lambda(x; 1) ,$$

which are the characters of  $\mathfrak{gl}_\infty$ -representations associated to partitions  $\lambda$ , with the orthogonality relation  $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda, \mu}$ . In this case the limiting inner product  $\langle -, - \rangle$  is the usual Hall inner product on  $\Lambda$  [70, Ch. I, §4]. The specialisation formula

$$s_\lambda(1, 1, \dots) = \prod_{s \in Y_\lambda} \frac{1}{h(s)} = \dim \lambda$$

is the dimension of the irreducible  $\mathfrak{gl}_\infty$ -representation with highest weight  $\lambda$ .

Now we turn to the general case. Since  $M_\lambda(x; q, q) = s_\lambda(x)$ , for  $\beta = 1$  we obtain again the expected basis of Schur functions  $U_\lambda(s; 1, k) = s_\lambda(x)$  for all  $k \geq 1$ . For generic  $\beta$ , we have

**Proposition B.8** *The rank  $k$  Uglov functions  $U_\lambda(x; \beta, k)$  have a unitriangular expansion in the basis of Schur functions  $s_\lambda(x)$  given by*

$$U_\lambda(x; \beta, k) = s_\lambda(x) + \sum_{\mu < \lambda} u_{\lambda, \mu}(\beta, k) s_\mu(x) \quad \text{with} \quad u_{\lambda, \mu}(\beta, k) \in \mathbb{C} .$$

**Proof:** Both the Schur and Uglov functions are limits of Macdonald functions and so can be expanded in unitriangular form by monomial symmetric functions. On the other hand, the bases  $\{m_\lambda\}$  and  $\{s_\lambda\}$  of  $\Lambda$  are related by a unimodular transformation (cf. (B.1) for  $t = q$ ) and so the monomial symmetric functions can be written in the form

$$m_\lambda(x) = s_\lambda(x) + \sum_{\mu < \lambda} a_{\lambda, \mu} s_\mu(x) .$$

With  $v_{\lambda, \mu}(\beta, k) := \lim_{p \rightarrow 1} v_{\lambda, \mu}(\omega p, \omega p^\beta)$ , this gives

$$\begin{aligned} U_\lambda(x; \beta, k) &= m_\lambda(x) + \sum_{\mu < \lambda} v_{\lambda, \mu}(\beta, k) m_\mu(x) \\ &= s_\lambda(x) + \sum_{\mu < \lambda} a_{\lambda, \mu} s_\mu(x) + \sum_{\mu < \lambda} v_{\lambda, \mu}(\beta, k) \left( s_\mu(x) + \sum_{\nu < \mu} a_{\mu, \nu} s_\nu(x) \right) \\ &= s_\lambda(x) + \sum_{\mu < \lambda} u_{\lambda, \mu}(\beta, k) s_\mu(x) \end{aligned}$$

with

$$u_{\lambda, \mu}(\beta, k) := a_{\lambda, \mu} + v_{\lambda, \mu}(\beta, k) + \sum_{\nu > \mu} v_{\lambda, \nu}(\beta, k) a_{\nu, \mu} ,$$

as required. ■

The orthogonality relation for the Uglov functions follows from that of the Macdonald functions (B.2) and one has

$$\langle U_\lambda, U_\mu \rangle_{\beta, k} = \delta_{\lambda, \mu} \prod_{\substack{s \in Y_\lambda \\ h(s) \equiv 0 \pmod k}} \frac{A(s) + 1 + \beta L(s)}{A(s) + \beta (L(s) + 1)} ,$$

where the limit of the Macdonald inner product is given by

$$\langle p_\lambda, p_\mu \rangle_{\beta, k} = \delta_{\lambda, \mu} z_\lambda \beta^{-\#\{\lambda_i \equiv 0 \pmod k\}} .$$

**Proposition B.9** *The rank  $k$  Uglov functions satisfy the bilinear sum relations*

$$\begin{aligned} \sum_{\lambda} \frac{1}{\langle U_{\lambda}, U_{\lambda} \rangle_{\beta, k}} U_{\lambda}(x; \beta, k) U_{\lambda}(y; \beta, k) &= \prod_{i, j \geq 1} \frac{1}{(1 - x_i y_j)^{\frac{\beta-1+k}{k}}} \prod_{a=1}^{k-1} \frac{1}{(1 - \omega^a x_i y_j)^{\frac{\beta-1}{k}}} \\ &= \exp \left( \sum_{a=1}^{k-1} \sum_{n \equiv a \pmod k} \frac{1}{n} p_n(x) p_n(y) + \frac{\beta}{k} \sum_{n=1}^{\infty} \frac{1}{n} p_{nk}(x) p_{nk}(y) \right). \end{aligned}$$

**Proof:** For the first equality, we first prove it when  $\beta \equiv 1 \pmod k$  and then analytically continue to arbitrary values. To this end we set  $\beta = sk + 1$  for  $s \in \mathbb{Z}_{>0}$ . Setting  $q = \omega p$  and  $t = \omega p^{\beta}$  in the infinite product appearing in the generalized Cauchy–Binet formula (B.6) for the Macdonald functions and then taking the limit  $p \rightarrow 1$ , we obtain

$$\begin{aligned} \frac{(tz; q)_{\infty}}{(z; q)_{\infty}} &= \prod_{n=0}^{\infty} \frac{1 - tz q^n}{1 - z q^n} \\ &= \prod_{n=0}^{\infty} \frac{1 - z \omega^{n+1} p^{n+\beta}}{1 - z \omega^n p^n} \\ &= \prod_{a=0}^{k-1} \prod_{m=0}^{\infty} \frac{1 - z \omega^{a+1} p^{mk+a+\beta}}{1 - z \omega^a p^{mk+a}} \\ &= \prod_{m=0}^{\infty} \frac{1 - p^{(m+s+1)k} z}{1 - p^{mk} z} \prod_{a=1}^{k-1} \frac{1 - \omega^a p^{(m+s)k+a} z}{1 - \omega^a p^{mk+a} z} = \frac{1}{(1-z)^{s+1}} \prod_{a=1}^{k-1} \frac{1}{(1 - \omega^a z)^s} \end{aligned}$$

and the result now follows. For the second equality, we proceed similarly with the argument of the exponential function appearing in the generalized Cauchy–Stanley identity (B.7) for the Macdonald functions to obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} \frac{1-t^n}{1-q^n} p_n(x) p_n(y) &= \sum_{n=1}^{\infty} \frac{1}{n} \frac{1-\omega^n p^{\beta n}}{1-\omega^n p^n} p_n(x) p_n(y) \\ &= \sum_{a=1}^k \sum_{m=0}^{\infty} \frac{1}{mk+a} \frac{1-\omega^a p^{\beta(mk+a)}}{1-\omega^a p^{mk+a}} p_{mk+a}(x) p_{mk+a}(y) \\ &= \sum_{a=1}^{k-1} \sum_{m=0}^{\infty} \frac{1}{mk+a} p_{mk+a}(x) p_{mk+a}(y) \\ &\quad + \frac{\beta}{k} \sum_{m=0}^{\infty} \frac{1}{m+1} p_{(m+1)k}(x) p_{(m+1)k}(y) \end{aligned}$$

and the result follows. ■

## C $\mathcal{W}$ -algebras

Let  $\mathfrak{g}$  be a finite-dimensional simply-laced Lie algebra of rank  $N$ , and denote its Weyl group by  $W(\mathfrak{g})$ . The *affine Lie algebra*  $\widehat{\mathfrak{g}}$  associated to  $\mathfrak{g}$  is defined as a one-dimensional central extension

$$0 \longrightarrow \mathbb{C}c \longrightarrow \widehat{\mathfrak{g}} \longrightarrow \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \longrightarrow 0,$$



where  $c$  is the *central charge* and  $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$  is the associated infinite-dimensional *loop algebra*.

Let  $e$  be a nilpotent element in the Lie algebra  $\mathfrak{g}$ . By the Jacobson–Morozov theorem, it can be completed to an  $\mathfrak{sl}_2$ -triple  $(e, f, h)$ , i.e. there exist elements  $f, h \in \mathfrak{g}$  with the Lie brackets

$$[h, e] = 2e, \quad [h, f] = -2f \quad \text{and} \quad [e, f] = h.$$

The  $\mathfrak{sl}_2$ -triples correspond to embeddings of the Lie algebra  $\mathfrak{su}_2$  into  $\mathfrak{g}$ . For example, nilpotent elements  $e$  of  $\mathfrak{g} = \mathfrak{gl}_N$  are classified (up to conjugation) by their traceless Jordan normal form, which corresponds to a partition  $\lambda = (\lambda_1, \dots, \lambda_\ell) = (1^{m_1} 2^{m_2} \dots)$  of  $N$  where  $m_i$  is the number of regular  $i \times i$  nilpotent block matrices

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad (\text{C.1})$$

in the Jordan form; alternatively,  $m_i$  is the multiplicity of the irreducible  $i$ -dimensional representation in the decomposition of a representation of  $\mathfrak{su}_2$  on  $\mathbb{C}^N$ . The two extreme cases are the trivial embedding  $\mathfrak{su}_2 \rightarrow 0 \in \mathfrak{g}$ , where  $e = 0$  corresponds to the maximal partition  $\lambda = (1^N)$  of length  $N$ , and the principal embedding  $\mathfrak{su}_2 \subset \mathfrak{g}$ , where  $e = e_{\text{pr}}$  is the regular nilpotent matrix (C.1) in the  $N$ -dimensional irreducible representation of  $\mathfrak{su}_2$  corresponding to the trivial partition  $\lambda = (N)$  of length one.

Given a nilpotent element  $e \in \mathfrak{g}$ , one constructs from the affine algebra  $\widehat{\mathfrak{g}}$  a vertex operator algebra  $\mathcal{W}(\mathfrak{g}, e)$  by a method called *Drinfeld–Sokolov reduction*. The two limiting extremes

$$\mathcal{W}(\mathfrak{g}, e = 0) = \widehat{\mathfrak{g}} \quad \text{and} \quad \mathcal{W}(\mathfrak{g}, e_{\text{pr}}) =: \mathcal{W}(\mathfrak{g})$$

are of particular interest. The  $\mathcal{W}(\mathfrak{g})$ -algebra has Virasoro quasi-primary fields

$$W_{d_a}(z) = \sum_{n \in \mathbb{Z}} \frac{W_{d_a, n}}{z^{d_a + n}} dz^{d_a}$$

of dimension  $d_a$  for  $a = 1, \dots, N$ , where  $z \in \mathbb{C}$ ,  $W_{d_a, n}$  are certain operators and  $d_a - 1$  is the  $a$ -th *exponent* of the Lie algebra  $\mathfrak{g}$  which satisfies the dimension formulas

$$\#\mathcal{W}(\mathfrak{g}) = \prod_{a=1}^N d_a \quad \text{and} \quad \dim \mathfrak{g} = \sum_{a=1}^N (2d_a - 1).$$

Note that  $d_a = a$  for  $\mathfrak{g} = \mathfrak{gl}_N$ .

The  $\mathcal{W}(\mathfrak{g})$ -algebras are generalizations of the Heisenberg and Virasoro algebras:  $\mathcal{W}(\mathfrak{gl}_1) = \widehat{\mathfrak{gl}}_1$  is the Heisenberg algebra  $\mathfrak{h}$  which is defined by generators  $\alpha_m$ ,  $m \in \mathbb{Z} \setminus \{0\}$  with the Heisenberg commutation relations

$$[\alpha_m, \alpha_n] = m \delta_{m+n, 0} c.$$

The  $\mathcal{W}(\mathfrak{sl}_2)$ -algebra is the Virasoro algebra which is defined by generators  $L_m$ ,  $m \in \mathbb{Z}$  with the relations

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{m}{12} (m^2 - 1) \delta_{m+n, 0} c.$$

In the general case,  $\mathcal{W}(\mathfrak{g})$  is a vertex algebra which does not admit a presentation in terms of generators and relations.

There is a surjective functor from the category of irreducible highest weight representations  $\mathcal{L}_\lambda$  of the affine Lie algebra  $\widehat{\mathfrak{g}}$ , which are parameterized by highest weights  $\lambda$  in the Cartan subalgebra of  $\mathfrak{g}$ , to the category of irreducible highest weight representations  $\mathcal{W}_\lambda$  of  $\mathcal{W}(\mathfrak{g}, e)$ , which are called *Verma modules*. In particular, this functor sends the vacuum representation  $\mathcal{L}_0$  to the vacuum representation  $\mathcal{W}_0$ .

Let  $h$  denote the dual Coxeter number of  $\mathfrak{g}$  ( $h = N$  for  $\mathfrak{g} = \mathfrak{gl}_N$ ).

**Definition C.2** A vector  $\psi \in \mathcal{W}_\lambda$  is called a Whittaker vector if

$$W_{d_a, n} \psi = 0 = W_{h, m} \psi \quad \text{and} \quad W_{h, 1} \psi = \psi$$

for all  $d_a \neq h$ ,  $a = 1, \dots, N$ ,  $n \geq 1$  and  $m > 1$ .

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